NON-ABELIAN ELECTRIC-MAGNETIC SYMMETRY AND FUSION RULES FOR MONOPOLES AND DYONS

NON-ABELIAN ELECTRIC-MAGNETIC SYMMETRY AND FUSION RULES FOR MONOPOLES AND DYONS

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CONTENTS

CHAPTER 1

INTRODUCTION

In the 1970s Goddard, Nuyts and Olive were the first to write down a rough version of what has become one of the most celebrated dualities in high energy physics. Following earlier work of Englert and Windey on the generalised Dirac quantisation condition [1] they showed that the charges of monopoles in a theory with gauge group G take values in the weight lattice of the dual gauge group G^*, now known as the GNO or Langlands dual group. Based on this fact they came up with a bold yet attractive conjecture: monopoles transform as representations of the dual group [2].

Within a year Montonen and Olive observed that the Bogomolny Prasad Sommerfield (BPS) mass formula for dyons [3, 4] is invariant under the interchange of electric and magnetic quantum numbers if the coupling constant is inverted as well [5]. This led to the dramatic conjecture that the strong coupling regime of some suitable quantum field theory is described by a weakly coupled theory with a similar Lagrangian but with the gauge group replaced by the GNO dual group and the coupling constant inverted. Moreover they proposed that in the BPS limit of a gauge theory where the gauge group is spontaneously broken to $U(1)$ the 't Hooft-Polyakov solutions [6, 7] in the original theory correspond to the heavy gauge bosons of the dual theory. Supporting evidence for the idea of viewing the 't Hooft-Polyakov monopoles as fundamental particles came from Erick Weinberg's zero-mode analysis in [8].

Soon after Montonen and Olive proposed their duality, Osborn noted that $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) would be a good candidate to possess the duality since BPS monopoles fall into the same BPS supermultiplets as the elementary particles of the theory [9]. $\mathcal{N} = 2$ SYM on the other hand has always been considered an unlikely candidate because the BPS monopoles fall into BPS multiplets that do not correspond to the elementary fields of the $\mathcal{N} = 2$ Lagrangian. In particular there are no semi-classical monopole states with spin equal to 1 so that the monopoles cannot be identified with

heavy gauge bosons. Most surprisingly the Montonen-Olive conjecture has never been proven for $\mathcal{N} = 4$ SYM whereas a different version of the duality has explicitly been shown to occur for the $\mathcal{N}=2$ theory in 1994 by Seiberg and Witten [10].
These authors started out from $\mathcal{N}=2$ SYM with the $SU(2)$ gauge group broken down to $U(1)$ and computed the exact effective Lagrangian of the theory to find a strong coupling phase described by SQED except that the electrons are actually magnetic monopoles. Moreover by softly breaking $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry they were able to show that in this strong coupling phase the monopoles condense and thereby demonstrated the 't Hooft-Mandelstam confinement scenario [11, 12]. Similar results hold for higher rank gauge groups broken down to their maximal abelian subgroups [13, 14]. In these cases we indeed have an explicit realisation of a magnetic abelian gauge group at strong coupling.

A fascinating aspect of Seiberg-Witten theory is that it does give rise to not just one strong coupling phase, but to several. In general only one of these phases contains massless monopoles while the other have an effective description in terms of dyons. A priori it is thus not clear what dynamically the relevant degrees of freedom are. This illustrates the necessity of a proper kinematic description of all possible degrees of freedom contained in the theory. For abelian phases it is not too hard to provide such a kinematic description while for non-abelian phases, that is a phases where the gauge group is broken down to a non-abelian subgroup, this problem has never been solved satisfactorily. The main challenge is to give a proper labelling of monopoles and dyons. In this thesis we tackle this issue.

As far as it concerns monopoles one finds classically that magnetic charges take value in the weight lattice of the dual group. Yet, there is no obvious rule to order these weights into irreducible representations with the appropriate dimensions and degeneracies, let alone that there is a manifest action of the dual group on the classical field configurations. To illustrate this we consider an example with gauge group $U(2)$ embedded as a subgroup in $SU(3)$. The magnetic charge lattice in this case corresponds to the root lattice of $SU(3)$ as depicted in figure 1.1. The GNO dual group of $U(2)$ is again isomorphic to $U(2)$, in other words the magnetic charge lattice can be identified with the weight lattice of $U(2)$. This group is by definition equal to $(U(1) \times SU(2))/\mathbb{Z}_2$. The $SU(2)$ weights can be identified with the components of the charges along the axis defined by one of the simple roots of $SU(3)$, say α_1. The $U(1)$-charges then correspond to the components of the charge along the axis perpendicular to α_1.
As a next step one would want to use the magnetic charge lattice to characterise the magnetic $U(2)$ multiplets as in accordance with the GNO conjecture. The origin of the charge lattice, i.e. the vacuum, can consistently be identified with the trivial representation of $U(2)$. Naively one would simply associate the doublet representation with unit $U(1)$ with the pair of weights $g_1 = \alpha_1 + \alpha_2$ and $g_2 = \alpha_2$. This relation, however, raises some questions about the action of the dual group which become even more pressing as soon

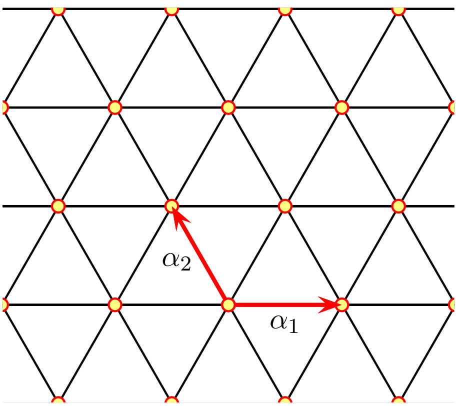


Figure 1.1: *The magnetic charge lattice for $G = U(2)$ corresponds to the root lattice of $SU(3)$ but also to the weight lattice of $U(2)$, hence G^* equals $U(2)$ in this case. One of the simple roots of $SU(3)$, say α_1 is identified with the root of $SU(2) \subset U(2)$.*

as one takes fusion of monopoles into account. There is an action that maps g_1 to g_2 and vice versa, suggesting that we are indeed dealing with a doublet. This action corresponds precisely to the action of the Weyl group $\mathbb{Z}_2$ of $U(2)$ generated by the refection in the line perpendicular to α_1. If we now consider the product of two monopoles in the doublet representation of the dual group then we expect from the GNO conjecture that one would obtain a singlet and a triplet. On the other hand in the classical theory the charge of a combined monopole equals the sum of charges of the constituents, and in this particular case thus equals $2g_1, g_1 + g_2$ or $2g_2$. The charges $2g_1$ and $2g_2$ are again related by the action of the Weyl group, but this Weyl action does not relate these two charges to $g_1 + g_2$. Moreover it is not clear if a combined classical monopole solution with charge $g_1 + g_2$, i.e. with an $SU(2)$ weight equal to zero, corresponds to a triplet or a singlet state.
One possible argument to resolve this issue comes from the fact that the action of the Weyl group is nothing but a large gauge transformation which suggests that charges on a single Weyl orbit should not distinguished as different weights of a dual representations. Instead such magnetic charges should be identified in the sense that they constitute a single gauge invariant charge sector. Pushing this argument a bit further one may conclude that a monopole should be labelled by an integral dominant weight, i.e. by the highest weight of an irreducible representations of the GNO dual group. The drawback of this interpretation is that it does neither manifestly show the dimension of the dual representations in the magnetic charge lattice nor does it directly explain the degeneracies implied by the fusion rules of G^*. From this example we conclude that solving this labelling problem for monopoles is closely related to proving the GNO conjecture. It is also important to note that the heuristic arguments above do not disprove the GNO conjecture since, as we have learned from Montonen and Olive, one does only expect the dual symmetry to be

manifest at strong coupling. From that perspective it is not very surprising that the dual symmetry is to a certain extent hidden in the classical regime. Nonetheless it should be clear that that the charge labels of monopoles are related to weights or dominant integral weights of the GNO dual group.
For dyons with non-vanishing electric and magnetic charges the situation is worse since, as we shall explain below, it is not known what the relevant algebraic object is that will give rise to a proper labelling.

Before we continue one should wonder whether Yang-Mills theories can have non-abelian phases at strong coupling. Both the classical $\mathcal{N}=4$ and $\mathcal{N}=2$ pure SYM theories have a continuous space of ground states corresponding to the vacuum expectation value of the adjoint Higgs field. A non-abelian phase corresponds to the Higgs VEV having degenerate eigenvalues. In the $\mathcal{N}=4$ theory the supersymmetry is sufficient to protect the classical vacuum structure even non-perturbatively [15]. So the non-abelian phases manifestly realised in the classical regime must survive at strong coupling as well. In the $\mathcal{N}=2$ theory the vacuum structure is changed in quite a subtle way by non-perturbative effects. In those subspaces of the quantum moduli space where a non-abelian phase might be expected there are no massless W-bosons. Instead the perturbative degrees of freedom correspond to photons and massless monopoles carrying abelian charges. In the best case there are some indications that a non-abelian phase may exist at strong coupling in certain $\mathcal{N}=2$ theories with a sufficient number of hyper multiplets [16, 17].

Although a kinematic description of monopoles and dyons in non-abelian phases is probably not very relevant for $\mathcal{N}=2$ SYM it may be important in understanding strongly coupled non-abelian phases of $\mathcal{N}=4$ SYM or non-abelian phases of other theories. We shall therefore discuss these issues in a general context. In a very specific theory, however, progress has already been made.
Quite recently Witten and Kapustin have found extraordinary new evidence to support the non-abelian Montonen-Olive conjecture. This evidence was constructed in an effort to show that the mathematical concept of the geometric Langlands correspondence arises naturally from electric-magnetic duality in physics [18].
The starting point for Kapustin and Witten is a twisted version of $\mathcal{N}=4$ gauge theory. They identify 't Hooft operators, which create the flux of Dirac monopoles, with Hecke operators. The labels of these operators are given by the generalised Dirac quantisation rule and can up to a Weyl transformation be identified with dominant integral weights of the dual gauge group. Since a dominant integral weight is the highest weight of a unique irreducible representation, magnetic charges thus correspond to irreducible representations of the dual gauge group. The moduli spaces of the singular BPS monopoles are identified with the spaces of Hecke modifications. The operation of bringing two separated monopoles together defines a non-trivial product of the corresponding moduli spaces. The resulting space can be stratified according to its singularities. Each singular

subspace is again the compactified moduli space of a monopole related to an irreducible representation in the tensor product. The multiplicity of the BPS saturated states for each magnetic weight is found by analysing the ground states of the quantum mechanics on the moduli space. The number of ground states given by the De Rham cohomology of the moduli space agrees with the dimension of the irreducible representation labelled by the magnetic weight. Moreover Kapustin and Witten exploited existing mathematical results on the singular cohomology of the moduli spaces to show that the products of 't Hooft operators mimic the fusion rules of the dual group. The operator product expansion (OPE) algebra of the 't Hooft operators thereby reveals the dual representations in which the monopoles transform.

There is an enormous amount of evidence to support the Montonen-Olive conjecture for the ordinary $\mathcal{N}=4$ SYM theory, see for example [19, 20, 21]. These results which mainly concern the invariance of the spectrum do not leave much room to doubt that the strongly coupled theory can be described in terms of monopoles. However, they do not say much about the fusion rules of these monopoles. If the original GNO conjecture does indeed apply for $\mathcal{N}=4$ SYM theory with residual non-abelian gauge symmetry, smooth monopoles should have properties similar to those of the singular BPS monopoles in the Kapustin-Witten setting. By the same token we claim that one can exploit these properties to find new evidence for the GNO duality in spontaneously broken theories. In chapter 3 we aim to set a first step in this direction by generalising the classical fusion rules found by Erick Weinberg for abelian BPS monopoles [22] to the non-abelian case. Our results indicate that smooth BPS monopoles are naturally labelled by integral dominant weights of the residual dual gauge group.

A stronger version of the GNO conjecture is that a gauge theory has a hidden electric-magnetic symmetry of the type $G\times G^*$. The problem with this proposal is that the dyonic sectors do not respect this symmetry in phases where one has a residual non-abelian gauge symmetry. In such phases it may be that in a given magnetic sector there is an obstruction to implement the full electric group. In a monopole background the global electric symmetry is restricted to the centraliser in G of the magnetic charge [23, 24, 25, 26, 27, 28]. Dyonic charge sectors are thus not labelled by a $G \times G^*$ representation but instead (up to gauge transformations) by a magnetic charge and an electric centraliser representation. For example in the case of $G = U(2)$, the centraliser for the magnetic charge α_2, see figure 1.1, equals the abelian subgroup $U(1) \times U(1)$. Hence, a dyon with this magnetic charge has an electric label corresponding to a representation of this abelian centraliser. For a dyon with magnetic charge equal to $\alpha_1 + 2\alpha_2$ the electric charge corresponds to a representation of the non-abelian centraliser group $U(2)$. This interplay of electric and magnetic degrees of freedom lacks in the $G \times G^*$ structure. Therefore one would like to find a novel algebraic structure reflecting this complicated pattern of the different electric-magnetic sectors in such a non-abelian phase. We see that one does not need this algebraic structure just to find a labelling for dyons but actually, first, to proof the consistency of the labelling already proposed, and second, to retrieve the fusion rules of non-abelian dyons which are not known up to today. In terms of centraliser representations one seems to run into trouble as soon as one considers fusion of dyons. On the electric side it is not clear how to define a tensor product involving the representations of distinct centraliser groups such as for example $U(1) \times U(1)$ and $U(2)$, even though the fusion rules for each of the centraliser groups are known. The algebraic structure we seek would thus have to generate the complete set of fusion rules for all the different sectors and in particular it would have to combine the different centraliser groups that may occur in such phases within one framework. It also has to be consistent with the fact that in the pure electric sector charges are labelled by the the full electric gauge group G, while in the purely magnetic sector, at least for the twisted $\mathcal{N} = 4$ theory considered by Kapustin and Witten in [18], monopoles form representations of the magnetic gauge group G^*.

In chapter 4 we propose a formulation of a gauge theory, based on the so-called skeleton group S. This is in general a non-abelian group that allows to manifestly include non-abelian electric and magnetic degrees of freedom. The skeleton group therefore implements (at least part of) the hidden electric-magnetic symmetry explicitly and the representation theory of S provides us with a consistent set of fusion rules for the dyonic sectors for an arbitrary gauge group. Nonetheless it does not quite fulfill our original objective. The skeleton group has roughly the product structure $S = \mathcal{W} \ltimes (T \times T^*)$ where T and T^* are the maximal tori of G and G^* and $\mathcal{W}$ the Weyl group. Therefore S contains neither the full electric gauge group G nor the magnetic group G^*, and this of course implies that its representation theory will not contain the representation theories of either G or G^*. We show, however, that in for example the purely electric sector the representation theory of the skeleton group is consistent with the representation theory of G.

The appearance of the skeleton group can be understood from gauge fixing and in that sense our approach matches an interesting proposal of 't Hooft [29]. In order to get a handle on non-perturbative effects in gauge theories, like chiral symmetry breaking and confinement, 't Hooft introduced the notion of non-propagating gauges. An important example of such a non-propagating gauge is the so-called abelian gauge. In this gauge a non-abelian theory can be interpreted as an abelian gauge theory with monopoles in it. This has led to a host of interesting approximation schemes to tackle the aforementioned non-perturbative phenomena which remain elusive from a first principle point of view up to today.
We present a generalisation of 't Hooft's proposal from an abelian to a minimally non-abelian scheme. That is where the skeleton group comes in. The attractive feature is that our generalisation does not affect the continuous part of the residual gauge symmetry after fixing, it is still abelian, but it adds (non-abelian) discrete components. That implies

that the non-abelian features of the effective theory manifest themselves through topological interactions only and that makes them manageable. The effective theories we end up with are actually (non-abelian) generalisations of Alice electrodynamics [30, 31, 32]. In this sense the effective description of the non-abelian theory with gauge group G in the skeleton gauge is an intricate merger of an abelian gauge theory and a (non-abelian) discrete gauge theory [33, 34]. Moreover, the skeleton gauge incorporates configurations which are not accesible in the abelian gauge. Hence, compared to the abelian gauge, the skeleton gauge and thereby the skeleton group may yield a much wider scope on certian non-perturbative features of the original gauge theory.

The motivation for exploring non-propagating gauges is to obtain a formulation of the theory as much as possible in terms of the physically relevant degrees of freedom. In that sense 't Hooft's approach looks like studying the Higgs phase in a unitary gauge, but it goes beyond that because one does not start out from a given phase determined by a suitable (gauge invariant) order parameter. Instead the effective theory in the abelian gauge is obtained after integrating out the non-abelian gauge field components. Nonetheless the resulting theory is particularly suitable to describe the Coulomb phase where the residual gauge symmetry is indeed abelian. Similarly, the skeleton group is related to a generalised Alice phase.

Once this gauge-phase relation is understood our skeleton formulation not only allows us to obtain the precise fusion rules for the mixed and neutral sectors of the theory, but as a bonus allows us to analyse the phase structure of gauge theories. Yang-Mills theories give rise to confining phases, Coulomb phases, Higgs phases, discrete topological phases, Alice phases etc. These phases not only differ in their particle spectra but also in their topological structure. It is therefore crucial to have a formulation that highlights the relevant degrees of freedom which allow one to understand what the physics of such phases is.

Starting from the skeleton gauge we are in a position to answer kinematic questions concerning different phases and possible transitions between them. For this purpose it is of utmost importance to work in a scheme that allows one to compute the fusion rules involving electric, magnetic and dyonic sectors. This is deduced from some common wisdom concerning the abelian case where the fusion rules are very simple: if there is a condensate corresponding to a particle with a certian electric or magnetic charge then any particle with a multiple of this charge will also be condensed. If two electric-magnetic charges confine then the sum of these charges will also confine. Given the fusion rules predicted by the skeleton group we can in priciple analyse all phases that emerge from generalised Alice phases by condensation or confinement.

CHAPTER 2

CLASSICAL MONOPOLE SOLUTIONS

This preliminary chapter serves multiple purposes. First, we want to explain what monopoles are and review some of their properties. Most of these are well known, a few are not. Second, we want to introduce some conventions, concepts and quantities that will be used in the remainder of this thesis. Finally, we want to explain how one can create some order in the monopole jungle by introducing several types of monopoles.

Very roughly speaking a monopole is a solution to equations of motion of a gauge theory with a non-vanishing magnetic charge. The nature of such a charge depends of course on exactly what Yang-Mills theory is considered and specifically what the gauge group is. Nonetheless, in general the magnetic charges constitute a discrete set which in turn and can be used to distinguish different monopoles within a given theory. These sets will be discussed in section 2.3. A cruder way to classify monopoles is to distinguish singular monopoles from smooth monopoles and non-BPS monopoles from BPS monopoles. In the first two sections of this chapter we shall review these properties and some related concepts.

2.1 SINGULAR MONOPOLES

Singular monopoles can appear in any gauge theory but the most basic example is a pure Yang-Mills theory. This can be either the abelian theory with gauge group that arises from the homogeneous Maxwell equations or a generalisation where the gauge group $U(1)$ is replaced by a larger and possibly non-abelian gauge group which we shall denote by H. The Lagrangian of such a Yang-Mills theory is completely defined in terms of the field strength tensor $F_{\mu\nu}$:

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{2.1}$$

The field strength tensor can be further expanded as $F_{\mu\nu} = F^a_{\mu\nu}t_a$, where t_a are the generators of the Lie algebra of H. In terms of the gauge field $A_\mu = A^a_\mu t_a$ we have

$$F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu - ie[A_\mu, A_\nu]. \tag{2.2}$$

Using differential forms one can write $A = A_\mu dx^\mu$ and $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ so that by definition $F = dA - ieA \wedge A$.
The equations of motion derived from the Lagrangian in (2.1) are given by:

$$\begin{aligned} D * F &= 0 \\ DF &= 0. \end{aligned} \tag{2.3}$$

The first of these two equations is the true equation of motion, the second is the Bianchi identity, see e.g. section 10.3 of [35]. The electric and and magnetic fields can be expressed in terms of the field strength tensor as

$$E^i = F^{0i} = -F^{i,0} = F_{i,0} \tag{2.4}$$

$$B^i = \frac{1}{2}\epsilon^{ijk}F_{ij} \iff F^{ij} = \epsilon^{ijk}B^k. \tag{2.5}$$

If the electric field vanishes we thus have

$$F = *B. \tag{2.6}$$

where $*$ corresponds to the Hodge star of the 3-dimensional Euclidean space $\mathbb{R}^3$.

A Dirac monopole [36] is a configuration of the electric-magnetic field with everywhere vanishing electric field and a static magnetic field of the form

$$B = \frac{G_0}{4\pi r^2}dr. \tag{2.7}$$

Note that for an abelian theory B is gauge invariant. If the gauge group is truly non-abelian the magnetic field transforms as

$$B \mapsto \mathcal{G}^{-1}B\mathcal{G} \tag{2.8}$$

under a gauge transformation

$$A \mapsto \mathcal{G}^{-1}\left(A + \frac{i}{e}d\right)\mathcal{G}. \tag{2.9}$$

Hence in a non-abelian theory the magnetic field of a Dirac monopole is defined by (2.7) up to gauge transformations.

From equation (2.7) we find for the field strength

$$F = *\left(\frac{G_0}{4\pi r^2}dr\right) = \frac{G_0}{4\pi}\sin\theta d\theta \wedge d\phi. \tag{2.10}$$

We shall check that this satisfies the equations motion (2.3) except at the origin where the Bianchi identity is violated. Note that since the field strength transforms in the adjoint representation of the gauge group, its covariant derivatives contain a commutator term with the gauge field. However, there is a gauge in which (2.7) is satisfied and in which the gauge field commutes with the field strength so that effectively the equations of motion reduce to the abelian case where the covariant derivatives of the field strength become ordinary derivatives. If the electric field vanishes such that $F = *B$ the equations of motion simplify to:

$$\begin{aligned} dB = 0 &\iff \epsilon^{ijk}\partial_j B_k = 0 \\ d*B = 0 &\iff \partial_i B_i = 0. \end{aligned} \tag{2.11}$$

While the curl of the magnetic field given in (2.7) obviously vanishes everywhere the divergence vanishes only away from the origin. As a matter fact fact one finds

$$\partial_i B_i = G_0 \delta^{(3)}(\mathbf{r}). \tag{2.12}$$

From Gauss' theorem we now see that the magnetic charge of the monopole equals G_0. Finally, making a comparison with (2.11) one finds that a monopole with non-vanishing charge G_0 violates the Bianchi identity at the origin. In that sense the Dirac monopole is singular at the origin.

Another way to view the singularity of the Dirac monopole is to consider the gauge field itself. One possible solution for the gauge field that gives rise to equation 2.7 is given by

$$A_+ = \frac{G_0}{4\pi}(1-\cos\theta)d\varphi. \tag{2.13}$$

On the negative z-axis (including the origin) where $d\varphi$ diverges A_+ is singular. This Dirac string, however, is merely a gauge artifact as can be seen by adopting the Wu-Yang formalism [37]. One can introduce a second gauge potential

$$A_- = -\frac{G_0}{4\pi}(1+\cos\theta)d\varphi \tag{2.14}$$

which also gives rise to 2.7 and which is well defined everywhere on $\mathbb{R}_3$ except for the positive z-axis and the origin. One could also construct other gauges where the Dirac

string does not coincide with the positive or negative z-axis. Nonetheless in every gauge there is a singularity at the origin O for non-vanishing values of G_0. The two gauge potentials A_+ and A_- thus give a complete description.
In the region where they are both well-defined A_+ and A_- are related by a gauge transformation:

$$A_- = \mathcal{G}^{-1}(\varphi)\left(A_+ + \frac{i}{e}d\right)\mathcal{G}(\varphi). \tag{2.15}$$

One can check

$$\mathcal{G}(\varphi) = \exp\left(\frac{ie}{2\pi}G_0\varphi\right). \tag{2.16}$$

We thus see that a singular monopole in $\mathbb{R}^3$ with non-vanishing magnetic charge defines a non-trivial H-bundle on $\mathbb{R}^3\ \{O\}$ and hence a non-trivial bundle on each sphere centred at the origin. We shall discuss this further in section 2.3. Nonetheless we already note that the non-triviality of the H-bundle is closely related to the violation of the Bianchi identity at the origin. In the bundle description one quite literally excises the origin from $\mathbb{R}^3$. One might therefore be tempted to say that such monopoles cannot exist. On the other hand one can simply accept that the magnetic field has certain prescribed singularities.
Still, in some sense singular monopoles seem avoidable if one restricts the fields to be smooth everywhere. This restriction does not rule out the possibility of having classical monopole solutions. It is also possible to have soliton like monopoles, see e.g. [6, 7]. Such monopoles satisfy the equations of motion, including the Bianchi identity, everywhere on $\mathbb{R}^3$. Since $\mathbb{R}^3$ is contractible a smooth monopole is related to a trivial bundle. Nonetheless these smooth monopoles behave asymptotically as Dirac mopoles. In section 2.3 we shall explain this relation between singular and smooth monopoles in further detail.

2.2 BPS MONOPOLES

A very special subtype of monopoles are BPS monopoles which by definition satisfy the BPS equation discussed below. Examples of smooth solutions of the BPS equation for $SU(2)$ are the 't Hooft-Polyakov monopoles [6, 7]. Precisely these monopoles have been conjectured by Montonen and Olive to correspond to the heavy gauge bosons of the S-dual gauge theory [5]. Though we shall mainly focus on smooth monopoles in this thesis it should be noted that for singular monopoles only BPS monopoles have been shown to transform as representations of the dual gauge group by Kapustin and Witten [18]. This motivates why also for smooth monopoles one should work in the BPS limit to obtain some insight in for example the fusion rules monopoles.

Instead of giving a detailed description of BPS solutions we shall merely try to give an idea of the general context by introducing the BPS limit and by sketching the derivation of the BPS equations and the BPS mass formula [3, 4]. In section 2.3 we shall come back to the asymptotic behaviour of smooth BPS monopoles.

In general smooth monopoles exist in certain Yang-Mills-Higgs theories. Special cases are Grand Unified theories or Yang-Mills-Higgs theories embedded in a larger theory with extra fermionic fields such as for example a super Yang-Mills theory. The Lagrangian for the Yang-Mills-Higgs theory can be written as:

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\mathrm{Tr}(D_\mu\Phi D^\mu\Phi) - V(\Phi). \tag{2.17}$$

Unless stated otherwise we shall take V to be the Mexican hat potential given by

$$V(\Phi) = \lambda/4\left(|\Phi|^2 - |\Phi_0|^2\right)^2. \tag{2.18}$$

The energy functional for the Yang-Mills-Higgs theory for this theory is given by:

$$E[\Phi, A] = \int \frac{1}{2}|D_0\Phi|^2 + \frac{1}{2}|D_k\Phi|^2 + \frac{1}{2}|B_k|^2 + \frac{1}{2}|E_k|^2 + V(\Phi)\, d^3x. \tag{2.19}$$

To retrieve the Bogomolny equations one should restrict the Higgs field Φ to transform in the adjoint representations. One can now rewrite the total energy as [3, 38]:

$$\begin{aligned} E[\Phi, A] = {} & |\Phi_0|\left(Q_e \sin\alpha + Q_m \cos\alpha\right) + \\ & \int \frac{1}{2}|D_0\Phi|^2 + \frac{1}{2}|B_k - \cos\alpha D_k\Phi|^2 + \frac{1}{2}|E_k - \sin\alpha D_k\Phi|^2 + V(\Phi)\, d^3x. \end{aligned} \tag{2.20}$$

where q_e and q_m are the so-called total abelian electric and magnetic charge defined by

$$Q_e = \frac{1}{|\Phi_0|}\int_{S^2_\infty} dS_i \mathrm{Tr}(E_i\Phi) \tag{2.21}$$

$$Q_m = \frac{1}{|\Phi_0|}\int_{S^2_\infty} dS_i \mathrm{Tr}(B_i\Phi). \tag{2.22}$$

If we now take the BPS-limit by letting $\lambda \to 0$ while keeping Φ_0 fixed and set

$$\sin\alpha = \frac{Q_e}{(Q_e^2 + Q_m^2)^{1/2}} \qquad \text{and} \qquad \cos\alpha = \frac{Q_m}{(Q_e^2 + Q_m^2)^{1/2}}, \tag{2.23}$$

we find from (2.20) the following inequality for the energy:

$$E \geq |\Phi_0|\left(Q_e \sin\alpha + Q_m \cos\alpha\right) = |\Phi_0|\left(Q_e^2 + Q_m^2\right)^{1/2} = |\Phi_0||Q_e + iQ_m|. \tag{2.24}$$

This lower bound for the energy is known as the Bogolmolny bound and is satisfied when the fields satisfy the following field equations:

$$B_i = \cos\alpha D_i\Phi$$

$$E_i = \sin\alpha D_i \Phi \tag{2.25}$$

$$D_0\Phi = 0.$$

The BPS bound is very natural in supersymmetric Yang-Mills theories in the sense that it is satisfied if the gauge group is broken but the supersymmetry remains unbroken.

In the special case that the electric charge vanishes, i.e. $Q_e = 0$, and all fields are static these three equations reduce to the Bogomolny or BPS equation:

$$B_i = D_i\Phi. \tag{2.26}$$

A solution to this BPS equation is called a *BPS monopole*. In general a solution of the equations of motion satisfying the Bogomolny bound is called a BPS *dyon*. As for ordinary particles the energy of a BPS monopole or dyon is bounded from below by its rest mass. Therefore the right hand side of equation (2.24) is called the BPS mass formula. To obtain a more profound understanding of the BPS limit it is very convenient to re-express the BPS formula as

$$M = \left|\Phi_0 \cdot \left(e\lambda + \frac{4\pi i}{e} g\right)\right|. \tag{2.27}$$

The quantaties $(\lambda)_{i=1\cdots r}$ and $(g)_{i=1\cdots r}$ are the electric charge and the magnetic charge. To determine the allowed values of the electric charge λ is somewhat delicate, see [39] and references therein. There exist classical solutions for every value of the electric charge but in the semi-classical theory the electric charge must be quantised. Without going into details we note that in a gauge theory with gauge group G it is heuristically clear that λ takes value in the weight lattice of G and that this is at least consistent with the fact that the BPS mass formula reproduces the mass of the massive gauge bosons with charge α equal to a root of G, see e.g. [40]:

$$M_\alpha = e|\Phi_0 \cdot \alpha|. \tag{2.28}$$

The magnetic charge is also quantised, we shall discuss this in much more detail in section 2.3.

An interesting adaptation of the theory is obtained by turning on the θ parameter. This means that one adds to the Lagrangian the term:

$$-\frac{\theta e^2}{32\pi^2}\int \mathrm{Tr}(F * F). \tag{2.29}$$

By introducing the complex coupling parameter τ as

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{2.30}$$

the total Lagrangian can now be conveniently rewritten in a commonly used form as:

$$\mathcal{L} = -\frac{e^2}{32\pi}\text{Im}\left[\tau \text{Tr}(F_{\mu\nu} + i * F_{\mu\nu})(F^{\mu\nu} + i * F^{\mu\nu})\right] + \frac{1}{2}\text{Tr}(D_\mu \Phi D^\mu \Phi) - V(\Phi). \quad (2.31)$$

The additional term does not change the equations of motion since (2.29) can be written as a total derivative, see e.g. section 23.5 of [41]. Even though the classical physics is unchanged by turning on the θ-parameter, the quantum theory is affected in a subtle way via instanton effects. As shown by Witten [39] these instanton effects give rise to non-integral abelian electric charges in the sense that

$$|\Phi_0|Q_e = e\Phi_0 \cdot \lambda + \frac{\theta e^2}{8\pi^2}|\Phi_0|Q_m, \quad (2.32)$$

with λ taking value on the weight lattice of G. This shift in the abelian electric charge is called the Witten effect. For an arbitrary value of θ the BPS mass formula is given by

$$M = ||\Phi_0|Q_e + i|\Phi_0|Q_m| = \sqrt{\frac{4\pi}{\tau}}|\Phi_0 \cdot (\lambda + \tau g)|. \quad (2.33)$$

In section 4.5 we shall review the invariance under S-duality transformations of this BPS mass formula for dyons in a gauge theory with arbitrary gauge group.

2.3 MAGNETIC CHARGE LATTICES

In this section we describe and identify the magnetic charges for several classes of monopoles. We shall start with a review for Dirac monopoles, then continue with smooth monopoles in spontaneously broken theories. Specifically for adjoint symmetry breaking we shall explain how the magnetic charge lattice can be understood in terms of the Langlands or GNO dual group of either the full gauge group or the residual gauge group. This will finally culminate in a thorough description of the set of magnetic charges for smooth BPS monopoles.

Dirac monopoles can be described as solutions of the Yang-Mills equations with the property that they are time independent and rotationally invariant. More importantly they are singular at a point as discussed in section 2.1. As a direct generalisation of the Wu-Yang description of $U(1)$ monopoles [37], singular monopoles in Yang-Mills theory with gauge group H correspond to a connection on an H-bundle on a sphere surrounding the singularity. The H-bundle may be topologically non-trivial, but in addition the monopole connection equips the bundle with a holomorphic structure. The classification of monopoles in terms of their magnetic charge then becomes equivalent to Grothendieck's classification of H-bundles on $\mathbb{CP}^1$. As a result, the magnetic charge has topological and holomorphic components, both of which play an important role in this thesis.

A different class of monopoles is found from smooth static solutions of a Yang-Mills-Higgs theory on $\mathbb{R}^3$ where the gauge group G is broken to a subgroup H. Since $\mathbb{R}^3$ is contractible the G-bundle is necessarily trivial. Choosing the boundary conditions so that the total energy is finite while the total magnetic charge is nonzero one finds that smooth monopoles behave asymptotically as Dirac monopoles. Since the long range gauge fields correspond to the residual gauge group this gives a non-trivial H-bundle at spatial infinity. The charges of smooth monopoles in a theory with G spontaneously broken to H are thus a subset in the magnetic charge lattice of singular monopoles in a theory with gauge group H .
Finally one can restrict solutions to the BPS sector where the energy is minimal. Smooth BPS monopoles are solutions of the BPS equations and therefore automatically solutions of the full equations of motion of the Yang-Mills-Higgs theory. Thus the charges of BPS monopoles are in principle a subset of the charges of smooth monopoles. This subset is determined by the so-called Murray condition which we shall introduce below. We shall also define the fundamental Murray cone which is related to the set of magnetic charge sectors.

2.3.1 QUANTISATION CONDITION FOR SINGULAR MONOPOLES

The magnetic charge of a singular monopole is restricted by the generalised Dirac quantisation condition [1, 2]. This consistency condition can be derived from the bundle description [37]. One can work in a gauge where the magnetic field has the form

$$B = \frac{G_0}{4\pi r^2} dr, \tag{2.34}$$

with G_0 an element in the Lie algebra of the gauge group H. This magnetic field corresponds to a gauge potential given by:

$$A_\pm = \pm \frac{G_0}{4\pi} (1 \mp \cos\theta)\, d\varphi. \tag{2.35}$$

The indices of the gauge potential refer to the two hemispheres. On the equator where the two patches overlap the gauge potentials are related by a gauge transformation:

$$A_- = \mathcal{G}^{-1}(\varphi) \left(A_+ + \frac{i}{e} d \right) \mathcal{G}(\varphi). \tag{2.36}$$

One can check

$$\mathcal{G}(\varphi) = \exp\left(\frac{ie}{2\pi} G_0 \varphi \right). \tag{2.37}$$

One obtains similar transition functions for associated vector bundles by substituting appropriate matrices representing G_0. All such transition functions must be single valued.

In the Dirac picture this means that under parallel transport around the equator electrically charged fields should not detect the Dirac string. Consequently we find for each representation the condition:

$$\mathcal{G}(2\pi) = \exp(ieG_0) = \mathbb{I}, \tag{2.38}$$

where $\mathbb{I}$ is the unit matrix. To cast this condition in slightly more familiar form we note that there is a gauge transformation that maps the magnetic field and hence also G_0 to a Cartan subalgebra (CSA) of H. Thus without loss of generality we can take G_0 to be a linear combination of the generators (H_a) of the CSA in the Cartan-Weyl basis:

$$G_0 = \frac{4\pi}{e}\sum_a g_a \cdot H_a \equiv \frac{4\pi}{e} g \cdot H. \tag{2.39}$$

The generalised Dirac quantisation condition can now be formulated as follows:

$$2\lambda \cdot g \in \mathbb{Z}, \tag{2.40}$$

for all charges λ in the weight lattice $\Lambda(H)$ of H.

We thus see that the magnetic weight lattice $\Lambda^*(H)$ defined by the Dirac quantisation condition is dual to the electric weight lattice $\Lambda(H)$. Consider for example the case where H is semi-simple as well as simply connected so that the weight lattice $\Lambda(H)$ is generated by the fundamental weights $\{\lambda_i\}$. Then $\Lambda^*(H)$ is generated by the simple coroots $\{\alpha_i^* = \alpha_i/\alpha_i^2\}$ which satisfy:

$$2\alpha_i^* \cdot \lambda_j = \frac{2\alpha_i \cdot \lambda_j}{\alpha_i^2} = \delta_{ij}. \tag{2.41}$$

As observed by Englert and Windey and Goddard, Nuyts and Olive the magnetic weight lattice can be identified with the weight lattice of the GNO dual group H^*. For example if we take $H = SU(n)$ and define the roots of $SU(n)$ such that $\alpha^2 = 1$, we see that $\Lambda^*(SU(n))$ corresponds to the root lattice of $SU(n)$. The root lattice of $SU(n)$ on the other hand is precisely the weight lattice of $SU(n)/\mathbb{Z}_n$. In the general simple case $\Lambda^*(H)$ resulting from the Dirac quantisation condition is the weight lattice $\Lambda(H^*)$ of the GNO dual group H^* whose weight lattice is the dual weight lattice of H and whose roots are identified with the coroots of H [1, 2]. In addition the center and the fundamental group of H^* are isomorphic to respectively the fundamental group and the center of H. Note that for all practical purposes the root system of H^* can be identified with the root system of H where the long and short roots are interchanged.

We shall not repeat the proof of the duality of the center and the fundamental group, but we will sketch the proof of the fact that the root lattice of H^* is always contained in the magnetic weight lattice. Finally we sketch the generalisation to any connected compact Lie group.

If H is not simply-connected we have $H = \widetilde{H}/Z$ where $\widetilde{H}$ is the universal cover of H and $Z \subset Z(\widetilde{H})$ a subgroup in the center of $\widetilde{H}$. Since $\Lambda(H) \subset \Lambda(\widetilde{H})$ with $Z = \Lambda(\widetilde{H})/\Lambda(H)$ the Dirac quantisation condition (2.40) applied on H is less restrictive than the condition for $\widetilde{H}$. Moreover one can check [2]:

$$\Lambda^*(H)/\Lambda^*(\widetilde{H}) = \Lambda(\widetilde{H})/\Lambda(H). \tag{2.42}$$

This implies that the coroot lattice $\Lambda^*(\widetilde{H})$ of H is always contained in the magnetic weight lattice $\Lambda^*(H)$ of H and in particular that any coroot $\alpha^* = \alpha/\alpha^2$ with α a root H, is contained in $\Lambda^*(H)$.
Without much effort this property can be shown to hold for any compact, connected Lie group. Any such group H say of rank r can be expressed as:

$$H = \frac{U(1)^s \times K}{Z}, \tag{2.43}$$

where K is a semi-simple and simply connected Lie group of rank $r - s$. The CSA of H is spanned by $\{H_a : a = 1, \ldots, r\}$ where H_a with $a \leq s$ are the generators of the $U(1)$ subgroups and $\{H_b : s < b \leq r\}$ span the CSA of K. Any weight of H can be expressed as $\lambda = (\lambda_1, \lambda_2)$ where λ_1 is a weight of $U(1)^s$ and λ_2 is a weight of K. Finally one finds that a magnetic charge G_0 defined by

$$G_0 = \frac{4\pi}{e}\alpha_j^* \cdot H, \tag{2.44}$$

where α_j is any of the $r - s$ simple roots of H, satisfies the quantisation condition.

H	H^*
$SU(nm)/\mathbb{Z}_m$	$SU(nm)/\mathbb{Z}_n$
$Sp(2n)$	$SO(2n+1)$
$Spin(2n+1)$	$Sp(2n)/\mathbb{Z}_2$
$Spin(2n)$	$SO(2n)/\mathbb{Z}_2$
$SO(2n)$	$SO(2n)$
G_2	G_2
F_4	F_4
E_6	$E_6/\mathbb{Z}_3$
E_7	$E_7/\mathbb{Z}_2$
E_8	E_8

Table 2.1: *Langlands or GNO dual pairs for simple Lie groups.*

In this section we have identified the magnetic charge lattice of singular monopoles with

H	H^*
$(U(1) \times SU(n))/\mathbb{Z}_n$	$(U(1) \times SU(n))/\mathbb{Z}_n$
$U(1) \times Sp(2n)$	$U(1) \times SO(2n+1)$
$(U(1) \times Spin(2n+1))/\mathbb{Z}_2$	$(U(1) \times Sp(2n))/\mathbb{Z}_2$
$(U(1) \times Spin(2n))\mathbb{Z}_2$	$(U(1) \times SO(2n))/\mathbb{Z}_2$

Table 2.2: *Examples of Langlands or GNO dual pairs for some compact Lie groups.*

the weight lattice of the dual group H^* of the gauge group H. In table 2.1 and 2.2 some examples are given of GNO dual pairs of Lie groups. Table 2.1 is complete up to some dual pairs related to $Spin(4n)$ that are obtained by modding out non-diagonal Z_2 subgroups of the center $\mathbb{Z}_2 \times \mathbb{Z}_2$. The GNO dual groups for these cases can be found in [2]. In section 2.3.3 we shall briefly explain how the dual pairing in table 2.2 is determined.

The magnetic charge lattice contains an important subset which we shall need later on: even if one restricts G_0 to the CSA there is some gauge freedom left which corresponds to the action of the Weyl group. Modding out this Weyl action gives a set of equivalence classes of magnetic charges which are naturally labelled by dominant integral weights in the weight lattice of H^*.

2.3.2 Quantisation condition for smooth monopoles

Yang-Mills-Higgs theories have solutions that behave at spatial infinity as singular Dirac monopoles but which are nonetheless completely smooth at the origin. This is possible if one starts out with a compact, connected, semi-simple gauge group G which is spontaneously broken to a subgroup H. Since all the fields are smooth, the gauge field defines a connection of a principal G-bundle over space which we take to be $\mathbb{R}^3$. The Higgs field is a section of a the adjoint bundle. As $\mathbb{R}^3$ is contractible the principal G-bundle is automatically trivial, so Φ is simply a Lie-algebra valued function. We would like to impose boundary conditions for the Higgs field Φ and the magnetic field B at spatial infinity which ensure that the total energy carried by a solution of the Yang-Mills-Higgs equations is finite. To our knowledge the question of which conditions are necessary and sufficient has not been answered in general. Below we review some standard arguments, many of them summarised in [42].

We assume an energy functional for static fields of the usual form

$$E[\Phi, A] = \int \frac{1}{2}|D_k\Phi|^2 + \frac{1}{2}|B_k|^2 + V(\Phi)\, d^3x, \tag{2.45}$$

where $D_k = \partial_k - ieA_k$ is the covariant derivative with respect to the G-connection A, and the magnetic field is given by $-ieB_k = -\frac{1}{2}ie\epsilon_{klm}F_{lm} = \frac{1}{2}\epsilon_{klm}[D_l, D_m]$. The potential V is a G-invariant function on the Lie algebra of G whose minimum is attained for non-vanishing value of $|\Phi|$; the set of minima is called the vacuum manifold. The variational equations for this functional are

$$\epsilon_{klm}D_l B_m = ie[\Phi, D_k\Phi], \qquad D_k D_k \Phi = \frac{\partial V}{\partial \Phi}. \tag{2.46}$$

In order to ensure that solutions of these equations have finite energy we require the fields Φ and B_i to have the following asymptotic form for large r:

$$\begin{aligned} \Phi &= \phi(\hat{r}) + \frac{f(\hat{r})}{4\pi r} + \mathcal{O}\left(r^{-(1+\delta)}\right) \qquad r \gg 1 \\ B &= \frac{G(\hat{r})}{4\pi r^2}dr \ + \ \mathcal{O}\left(r^{-(2+\delta)}\right) \qquad r \gg 1. \end{aligned} \tag{2.47}$$

Here $\delta > 0$ is some constant and $\phi(\hat{r}), f(\hat{r})$, and $G(\hat{r})$ are smooth functions on S^2 taking values in the Lie algebra of the gauge group G which have to satisfy various conditions. First of all, the function ϕ has to take values in the vacuum manifold of the potential V. It is thus a smooth map from the two-sphere to that vacuum manifold. The homotopy class of that map defines the monopole's topological charge [42]. Since the vacuum manifold can be identified with the coset space G/H the topological charge takes value in $\pi_2(G/H)$. Secondly, writing ∇ for the induced exterior covariant derivative tangent to the two-sphere "at infinity" it is easy to check that

$$\nabla\phi = 0, \qquad \nabla f = 0 \tag{2.48}$$

are necessary conditions for the integral defining the energy (2.45) to converge. The first of these equations implies

$$[\phi(\hat{r}), G(\hat{r})] = 0. \tag{2.49}$$

The quickest way to see this is to note that the curvature on the two-sphere at infinity is

$$F^\infty = *\left(\frac{G(\hat{r})}{4\pi r^2}dr\right) = \frac{G(\hat{r})}{4\pi}\sin\theta d\theta \wedge d\varphi. \tag{2.50}$$

Since $[\nabla, \nabla] = -ieF^\infty$, it follows that $\nabla\phi = 0$ implies $[F^\infty, \phi] = 0$. Finally we also require that

$$\nabla G = 0, \tag{2.51}$$

and that

$$[\phi(\hat{r}), f(\hat{r})] = 0. \tag{2.52}$$

The condition (2.51) is crucial for what follows, and seems to be satisfied for all known finite energy solutions [42]. The condition (2.52) is required so that the first of the equations (2.46) is satisfied to lowest order when the expansion (2.47) is inserted. In general there will be additional requirements on the functions ϕ and f that depend on the precise form of the potential V in (2.45). Since we do not specify V we will not discuss these further.

The above conditions can be much simplified by changing gauge. The equations (2.48) and (2.51) imply that for each of the Lie-algebra valued functions ϕ, f and G the values at any two points on the two-sphere at infinity are conjugate to one another (the required conjugating element being the parallel transport along the path connecting the points). We can therefore pick a point $\hat{r}_0$, say the north pole, and gauge transform ϕ into $\Phi_0 = \phi(\hat{r}_0)$, f into $\Phi_1 = f(\hat{r}_0)$ and G into $G_0 = G(\hat{r}_0)$. However, since S^2 is not contractible, we will, in general, not be able to do this smoothly everywhere on the two-sphere at infinity. If, instead, we cover the two-sphere with two contractible patches which overlap on the equator, then there are smooth gauge transformations g_+ and g_- defined, respectively, on the northern and southern hemisphere, so that the following equations hold where they are defined:

$$\phi(\hat{r}) = g_\pm^{-1}(\hat{r})\Phi_0 g_\pm(\hat{r}) \tag{2.53}$$
$$f(\hat{r}) = g_\pm^{-1}(\hat{r})\Phi_1 g_\pm(\hat{r}) \tag{2.54}$$
$$G(\hat{r}) = g_\pm^{-1}(\hat{r})G_0 g_\pm(\hat{r}). \tag{2.55}$$

After applying these gauge transformation, our bundle is defined in two patches, with transition function $\mathcal{G} = g_+ g_-^{-1}$ defined near the equator. This transition function leaves Φ_0 invariant, and hence lies in the subgroup H of G which stabilises Φ_0. This, by definition, is the residual or unbroken gauge group referred to in the opening paragraph of this section. It follows from (2.49), that $[\Phi_0, G_0] = 0$, so that G_0 lies in the Lie algebra of H. Similarly, (2.52) implies that Φ_1 lies in the Lie algebra of H. After applying the local gauge transformations (2.53), the asymptotic form of the fields is

$$\begin{aligned} \Phi &= \Phi_0 + \frac{\Phi_1}{4\pi r} + \mathcal{O}\left(r^{-(1+\delta)}\right) \\ B &= \frac{G_0}{4\pi r^2} dr \ + \ \mathcal{O}\left(r^{-(2+\delta)}\right). \end{aligned} \tag{2.56}$$

Note that "the Higgs field at infinity" is now constant, taking the value Φ_0 everywhere. In particular, it therefore belongs to the trivial homotopy class of maps from the two-sphere to the vacuum manifold. The topological charges originally encoded in the map ϕ can no longer be computed from the Higgs field. Instead they are now encoded in transition function $\mathcal{G}$. Since, in the new gauge, the magnetic field at large r is that of a Dirac monopole with gauge group H we can relate the transition function to the magnetic charge as before:

$$\mathcal{G}(\varphi) = \exp\left(\frac{ie}{2\pi} G_0 \varphi\right) \tag{2.57}$$

We thus obtain a quantisation condition for the magnetic charge of smooth monopoles, following the same arguments as in the singular case. For each representation of H the gauge transformation must be single valued if one goes around the equator, so that

$$2\lambda \cdot g \in \mathbb{Z}, \tag{2.58}$$

for all charges λ in the weight lattice of H.

One observes that the magnetic charge lattice of smooth monopoles lies in the weight lattice of the GNO dual group H^*. There is, however, another consistency condition [1]. Note that a single valued gauge transformation on the equator defines a closed curve in H as well as in G, starting and ending at the unit element. Since the original G-bundle is trivial, this closed curve has to be contractible in G. Therefore the monopole's topological charge is labelled by an element in $\pi_1(H)$ which maps to a trivial element in $\pi_1(G)$. This is consistent with our earlier remark that the topological charge is an element of $\pi_2(G/H)$ because of the isomorphism $\pi_2(G/H) \simeq \ker(\pi_1(H) \to \pi_1(G))$.

To find the appropriate charge lattice we use the fact that a loop in G is trivial if and only if its lift to the universal covering group $\widetilde{G}$ is also a loop (closed path). This implies that for smooth monopoles the quantisation condition should not be evaluated in the group H itself but instead in the group $\widetilde{H} \subset \widetilde{G}$ defined by the Higgs VEV Φ_0. Consequently equation (2.58) must not only hold for all representations of H but in fact for all representations of $\widetilde{H}$. Note that if G is simply connected then $\widetilde{H} = H$. In the next section we shall work this topological condition out in more detail.

2.3.3 Quantisation condition for smooth BPS monopoles

In chapter 3 we will mainly focus on BPS monopoles in spontaneously broken theories. We shall therefore work out some results of the previous section in somewhat more detail for the BPS case. We shall also give an explicit description of the magnetic charge lattice. In addition we introduce terminology that is conveniently used in the remainder of this thesis.

By BPS monopoles we mean static, finite energy solutions of the BPS equations

$$B_i = D_i \Phi \tag{2.59}$$

in a Yang-Mills-Higgs theory with a compact, connected, semi-simple gauge group G. The equations (2.59) imply the second order equations (2.46). In order to obtain finite energy solutions we again impose the boundary conditions (2.47). As in the previous section we can gauge transform these into the form (2.56). There are some differences with the non-BPS case. The potential V in (2.45) vanishes in the BPS limit, so does not furnish any conditions on the functions ϕ and f. On the other hand, by substituting (2.56) in the BPS equation and solving order by order one finds that $f = -G$, or, equivalently, $\Phi_1 = -G_0$. As before we have $[\Phi_0, G_0] = 0$, so in the BPS case we automatically have $[\Phi_0, \Phi_1] = 0$. From now on we shall thus define a BPS monopole to be a smooth solution of the BPS equations satisfying the boundary condition (2.47) with $\Phi_1 = -G_0$. After applying the local gauge transformations discussed in the previous section, these boundary conditions are equivalent to

$$\begin{aligned} \Phi &= \Phi_0 - \frac{G_0}{4\pi r} + \mathcal{O}\left(r^{-(1+\delta)}\right) \\ B &= \frac{G_0}{4\pi r^2}\hat{r} \, + \, \mathcal{O}\left(r^{-(2+\delta)}\right), \end{aligned} \tag{2.60}$$

where Φ_0 and G_0 are commuting elements in the Lie algebra of G. These boundary conditions are sufficient to guarantee that the energy of the BPS monopole is finite. It is in general not known what the necessary boundary conditions are to obtain a finite energy configuration. It is expected though [43, 44], and true for $G = SU(2)$ [45], that the boundary conditions above follow from the finite energy condition and the BPS equation.

Before we give an explicit description of the magnetic charge lattice let us summarise some properties of the residual gauge group. Since $[\Phi_0, G_0] = 0$ there is a gauge transformation that maps Φ_0 and G_0 to our chosen CSA of G. Without loss of generality we can thus express Φ_0 and G_0 in terms of the generators (H_a) of that CSA:

$$\begin{aligned} \Phi_0 &= \ \mu \cdot H \\ G_0 &= \frac{4\pi}{e} g \cdot H. \end{aligned} \tag{2.61}$$

The residual gauge group is generated by generators L in the Lie algebra of G satisfying $[L, \Phi_0] = 0$. Since generators in the CSA by definition commute with the Higgs VEV the residual group H contains at least the maximal torus $U(1)^r \subset G$. For generic values of the Higgs VEV this is the complete residual gauge symmetry. If the Higgs VEV is perpendicular to a root α the residual gauge group becomes non-abelian. This follows from the action of the corresponding ladder operator E_α in the Cartan-Weyl basis on the Higgs VEV: $[E_\alpha, \Phi_0] = -\mu \cdot \alpha \, E_\alpha = 0$. Accordingly we shall call a root of G *broken* if it has a non-vanishing inner product with μ and we shall define it to be *unbroken* if this inner product vanishes.
The residual gauge group is locally of the form $U(1)^s \times K$, where K is some semi-simple Lie group. The root system of K is derived from the root system of G by removing the broken roots. Similarly, the Dynkin diagram of K is found from the Dynkin diagram of G by removing the nodes related to broken simple roots. For completeness we finally define a fundamental weight to be (un)broken if the corresponding simple root is (un)broken.

The magnetic charge lattice for smooth monopoles lies in the dual weight lattice of H, as we saw in the previous chapter. For adjoint symmetry breaking the weight lattice of H is isomorphic to the weight lattice of G. Moreover the isomorphism respects the action of the Weyl group $\mathcal{W}(H) \subset \mathcal{W}(G)$. The existence of an isomorphism between $\Lambda(G)$ and $\Lambda(H)$ is easily understood since the weight lattices of H and G are determined by the irreducible representations of their maximal tori which are isomorphic for adjoint symmetry breaking. A natural choice for the CSA of H is to identify it with the CSA of G. In this case $\Lambda(G)$ and $\Lambda(H)$ are not just isomorphic but also isometric. Since the roots of H can be identified with roots of G and since the Weyl group is generated by the reflections in the hyperplanes orthogonal to the roots, this isometry obviously respects the action of $\mathcal{W}(H)$. Often the CSA of H is identified with the CSA of G only up to normalisation factors. This leads to rescalings of the weight lattice of H. Of course one can apply an overall rescaling without spoiling the invariance of weight lattice under the Weyl reflections. One can also choose the generators of $U(1)^s$-factor such that the corresponding charges are either integral or half-integral. Note that these rescalings again respect the action of $\mathcal{W}(H)$. To avoid confusion we shall ignore these possible rescalings in the remainder of this thesis and take $\Lambda(H)$ to be isometric to $\Lambda(G)$.
Since the weight lattices $\Lambda(H)$ and $\Lambda(G)$ are isometric their dual lattices $\Lambda^*(H)$ and $\Lambda^*(G)$ are isometric too. We thus see that the Dirac quantisation condition (2.58) for adjoint symmetry breaking can consistently be evaluated in terms of either H or G.

Remember that for smooth monopoles there is yet another condition: since one starts out from a trivial G bundle the magnetic charge should define a topologically trivial loop in G as explained in the previous section. For general symmetry breaking this implies that the Dirac quantisation condition must be evaluated with respect to weight lattice of $\widetilde{H} \subset \widetilde{G}$, where $\widetilde{G}$ is the universal covering group of G. For adjoint symmetry breaking we can consistently lift the quantisation condition to G; the weight lattice of $\widetilde{H}$ is isometric to the weight lattice of $\widetilde{G}$. The weight lattice of $\widetilde{G}$ is generated by the fundamental weights $\{\lambda_i\}$ and hence the magnetic charge lattice for smooth BPS monopoles is given by the solutions of:

$$2\lambda_i \cdot g \in \mathbb{Z}, \tag{2.62}$$

for all fundamental weights λ_i of $\widetilde{G}$. The most general solution of this equation is easily solved in terms of the simple coroots of G:

$$g = \sum_i m_i \alpha_i^* \qquad m_i \in \mathbb{Z}, \tag{2.63}$$

with $\alpha_i^* = \alpha_i/\alpha_i^2$ and $\{\alpha_i\}$ the simple roots of G.
We thus conclude that the magnetic charge lattice for smooth BPS monopoles is generated by the simple coroots of G. The resulting coroot lattice $\Lambda^*(\tilde{G})$ corresponds precisely to the weight lattice $\Lambda(\widetilde{G}^*)$ of the GNO dual group $\widetilde{G}^*$ as mentioned in section 2.3.1.

Similarly, the dual lattice $\Lambda^*(\widetilde{H})$ can be identified with $\Lambda(\widetilde{H}^*)$. With $\Lambda^*(\widetilde{G})$ being isometric to $\Lambda^*(\widetilde{H})$ we now conclude that the weight lattice of $\widetilde{G}^*$ can be identified with the weight lattice of $\widetilde{H}^*$. For G simply connected we have thus established an isometry between the root lattice of G^* and the weight lattice of H^*. We have used this isometry to compute the GNO dual pairs given in table 2.2 which appear in the minimal adjoint symmetry breaking of the classical Lie groups.

Above we have seen that the magnetic charge lattice for smooth BPS monopoles corresponds to the coroot lattice of the gauge group G. One can split the set of coroots into broken coroots and unbroken coroots. A coroot is defined to be broken or unbroken if the corresponding root is respectively broken or unbroken. Note that the unbroken coroots are precisely the roots of H^*. The distinction between broken and unbroken applies in particular to simple coroots. There is, however, alternative terminology for the components of the magnetic charges that reflects these same properties. Broken simple coroots are identified with *topological charges* while unbroken simple coroots are related to so-called *holomorphic charges*.
Remember that the magnetic charge $g = m_i\alpha_i^*$ defines an element in $\ker(\pi_1(H) \to \pi_1(G))$. One might hope that every single magnetic charge g, i.e. every point in the coroot lattice, defines a unique topological charge. If in that case a static monopole solution does indeed exist even its stability under smooth deformations is guaranteed. Such a picture does hold for maximally broken theories where the residual gauge group equals the maximal torus $U(1)^r \subset G$. If H contains a non-abelian factor the situation is slightly more complicated because these factors are not detected by the fundamental group. For G equal to $SU(3)$ for instance the magnetic charge lattice is 2-dimensional and $\pi_1\left(SU(3)\right) = 0$. In the maximally broken theory we have $\pi_1(U(1) \times U(1)) = \mathbb{Z} \times \mathbb{Z}$, while for minimal symmetry breaking $\pi_1(U(2)) = \pi_1(U(1)) = \mathbb{Z}$. As a rule of thumb one can say that the components of the magnetic charges related to the $U(1)$-factors in H are topological charges. It should be clear that these components correspond to the broken simple coroots. We therefore call the coefficients $m_i = 2\lambda_i \cdot g$ with λ_i a broken fundamental weight the topological charges of g. The remaining components of g are often called holomorphic charges.

2.3.4 MURRAY CONDITION

We have found that magnetic charges of smooth monopoles in a Yang-Mills-Higgs theory lie on the coroot lattice of the gauge group. In the BPS limit there is yet another consistency condition which was first discovered by Murray for $SU(n)$ [46]. We refer to this condition as *the Murray condition* even though its final formulation for general gauge groups stems from a paper by Murray and Singer [44]. For a derivation of the Murray condition we refer to these original papers. We shall only briefly review some properties of roots which are crucial for the Murray condition. Next we shall formulate the results of Murray and Singer in such a way that the set of magnetic charges for BPS monopoles can easily be identified. Finally we show that our formulation is equivalent to the condition as stated in [44]. Both formulations of the Murray condition will show up in later sections. The set of magnetic charges satisfying the Murray condition shall be called the Murray cone. At the end of this section we shall also introduce the fundamental Murray cone.

The Murray condition hinges on the fact that one can split the root system of G into positive and negative roots with respect to the Higgs VEV. If for a root α we have $\alpha \cdot \mu > 0$ it is by definition positive and if $\alpha \cdot \mu < 0$ it is negative. The set of roots is now partitioned into two mutually exclusive sets, at least if the residual gauge group is abelian. In that case we can as usual define a simple root to be a positive root that cannot be expressed as a sum of two other positive roots and it turns out that the Higgs VEV defines a unique set of simple roots. These form a basis of the root diagram is such a way that every positive root is a linear combination of simple roots with positive coefficients and similarly every negative root is a linear combination with negative coefficients. In the non-abelian case there exist roots such that $\alpha \cdot \mu = 0$. Hence there are several choices for a set of simple roots which are consistent with the Higgs VEV. Again for a fixed choice such simple roots must by definition have the property that all roots are a linear combination of simple roots with either only positive or only negative coefficients. In addition the simple roots must have either a strictly positive or a vanishing inner product with the Higgs VEV:

$$\alpha_i \cdot \mu \geq 0. \tag{2.64}$$

This condition implies that μ must lie in the closure of the fundamental Weyl chamber. In the remainder of this thesis we shall always choose simple roots so that the inequality in (2.64) is satisfied.
All choices for a set of simple roots respecting the Higgs VEV are related by the residual Weyl group $\mathcal{W}(H)$. This is seen as follows. In general all choices of simple roots in the root system of G are related by the Weyl group $\mathcal{W}(G)$ of G. Since Weyl transformations are orthogonal we have for all $w \in \mathcal{W}(G)$ $w(\alpha_i) \cdot \mu = \alpha_i \cdot w^{-1}(\mu)$. Given a set of positive roots satisfying (2.64) the action of $w \in \mathcal{W}(G)$ gives another set of simple roots satisfying the same condition if and only if μ and $w(\mu)$ lie in the closure of same Weyl chamber. This is only possible if μ is actually invariant under w, implying that $w \in \mathcal{W}(H) \subset \mathcal{W}(G)$.

Above we have defined a positivity condition for the roots of G that is consistent with the Higgs VEV. This same definition is applicable for coroots since these differ from the roots by a scaling. We now also extend this definition of positivity in a consistent way to the complete (co)root lattice. We call an element on the (co)root lattice positive if it is a linear combination of simple (co)roots with positive integer coefficients. Note that the intersection of the set of positive elements in the (co)root lattice with the set of (co)roots is precisely the set of positive (co)roots. Finally we see that if the Higgs VEV lies in the fundamental Weyl chamber then the inner product of any positive element in the (co)root lattice with μ is non-negative.

Murray and Singer have found that the magnetic charge must be positive with respect to all possible choices of simple roots consistent with the Higgs VEV. This means that in the expansion $g = \sum_i m_i \alpha_i^*$ the coefficients m_i should be positive for all possible choices of simple roots (α_i) that satisfy $\alpha_i \cdot \mu \geq 0$. The Murray condition can be summarised as follows:

$$2w(\lambda_i) \cdot g \geq 0 \;\; \forall w \in \mathcal{W}(H), \; \forall \lambda_i. \tag{2.65}$$

This is seen from the fact that the fundamental weights and simple roots satisfy $2\lambda_i \cdot \alpha_j^* = \delta_{ij}$ and that all allowed choices of positive simple roots and fundamental weights are related by the residual Weyl group $\mathcal{W}(H) \subset \mathcal{W}(G)$.

The Murray condition defines a solid cone in the CSA. In combination with the Dirac quantisation condition this results in a discrete cone of magnetic charges. We shall call this cone the Murray cone. As an example one can consider $SU(3)$ broken to either $U(1) \times U(1)$ or $U(2)$ as depicted in figure 2.1. In the first case the Weyl group of the residual gauge group is trivial and the Murray condition simply implies that the topological charges must be positive. In the second case the residual Weyl group is $\mathbb{Z}_2$, the reflections in the line perpendicular to α_1. Consequently there are two possible choices of positive simple roots which makes the Murray condition more restrictive. The topological charge still has to be positive, just like the holomorphic charge, but the holomorphic charge is bounded by the topological charge.

We shall finish this section with yet another formulation of the Murray condition originating from proposition 4.1 in the paper of Murray and Singer [44]. It relies on the fact that the holomorphic charges can be minimised under the action of the residual Weyl group. For any element g in the coroot lattice there exists a uniquely determined reduced magnetic charge $\tilde{g}$ in the Weyl orbit of g such that $\alpha_j \cdot \tilde{g} \leq 0$ for all unbroken simple roots α_j. The Murray condition can be expressed in terms of this minimised charges. A magnetic charge g is positive with respect to any chosen set of simple roots if and only if for a fixed choice of simple roots its reduced magnetic charge is positive. The reduced magnetic charge should thus satisfy:

$$2\lambda_i \cdot \tilde{g} \geq 0 \;\; \forall \lambda_i. \tag{2.66}$$

We shall shortly show that $\tilde{g}$ does indeed exist and is unique. But already we can see that this last condition easily follows from (2.65). Since $\tilde{g} = \tilde{w}(g)$ for some $\tilde{w} \in \mathcal{W}(H)$ we have $w(\lambda_i) \cdot \tilde{g} = w(\lambda_i) \cdot \tilde{w}(g) = \tilde{w}^{-1}(w(\lambda_i)) \cdot g = w'(\lambda_i) \cdot g \geq 0$, where $w' = \tilde{w}^{-1} w \in \mathcal{W}(H)$. To show equivalence, however, we also have to show that (2.65) follows from (2.66), which boils down to proving the following proposition:

Proposition 2.1 *If the reduced magnetic charge $\tilde{g}$ is positive then $w(\tilde{g})$ is positive for all $w \in \mathcal{W}(H)$.*

Proof. We take the gauge group G broken to H. The magnetic charges of BPS monopoles lie on the coroot lattice of G or equivalently the root lattice of G^*. We can assume G to be simply-connected since this does not affect the magnetic charge lattice. Under this assumption there is an isomorphism λ from the coroot lattice $\Lambda^*(G)$ to the weight lattice $\Lambda(H^*)$ of H^* as discussed in section 2.3.3. Up to discrete factors H^* is of the form $U(1)^s \times K^*$, where K^* is some semi-simple Lie group. Similarly, the set of simple roots of G is split up into s broken roots $\{\alpha_i\}$ with $0 < i \leq s$ and $r - s$ unbroken roots $\{\alpha_j\}$ with $s < j \leq r$. The magnetic charges are thus expanded as $g = \sum_i m_i \alpha_i^* + \sum_j h_j \alpha_j^*$. The linear map λ is defined by the images of the simple coroots. For the unbroken simple coroots this is particularly simple. We have $\lambda(\alpha_j^*) = \alpha_j^*$. More generally the image is given in terms of the abelian charges and a weight of K^*. While the abelian charges are identified with the topological charges $\{m_i\}$ the non-abelian charge can be expanded in terms of the fundamental weights λ_j of K^*. The coefficients, i.e. the Dynkin labels, are given by the projection on the roots of K^*: $k_j = 2\alpha_j^* \cdot g/\alpha_j^{*2}$. Being sums of multiples of the entries of the Cartan matrix of G^* these labels are indeed integers.

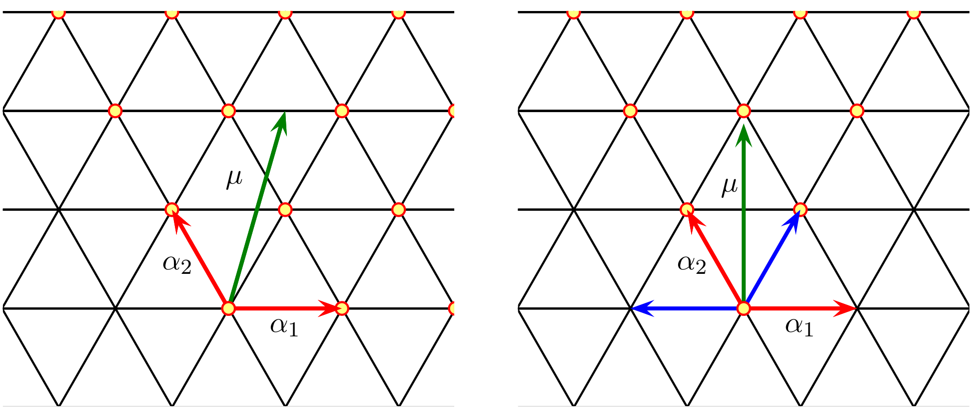


Figure 2.1: *The Murray cone for $SU(3)$ as a subset of the Cartan subalgebra. If the residual gauge group equals $U(1) \times U(1)$ (left) the Higgs VEV determines a unique set of simple roots. The static BPS monopoles have magnetic charges equal to a positive linear combination of these roots. These charges are in one-to-one correspondence with the positive topological charges. If the residual gauge group is $U(2)$ (right) there are two choices of simple roots. Only those charges that have a positive expansion for both these choices correspond to non-empty moduli spaces of static BPS monopoles. There is only a single topological charge which is proportional to the inner product of the magnetic charge with the Higgs VEV μ. As can been seen from the picture the total magnetic charge is not uniquely determined by the topological charge alone: non-abelian monopoles may carry non-trivial holomorphic charges.*

We can now easily prove that the reduced magnetic charge $\tilde{g}$ exists and is unique. Let $h := \lambda(g)$. Any weight $h \in \Lambda(H^*)$ can be mapped to a unique weight $\tilde{h}$ in the anti-fundamental Weyl chamber via a Weyl transformation. We thus have $\tilde{h} \cdot \alpha \leq 0$. The reduced magnetic charge $\tilde{g}$ is fixed by $\lambda(\tilde{g}) = \tilde{h}$. Since $2\lambda(g) \cdot \alpha_j^*/\alpha_j^{*2} = 2g \cdot \alpha_j^*/\alpha_j^{*2}$ we have $\alpha_j^* \cdot g \leq 0$ for all unbroken roots of G^*. The same inequality holds for the unbroken roots of G itself.

We now return to the proof of the proposition. First we shall use the fact that λ respects the residual Weyl group is the sense that $\lambda\left(w(g)\right) = w\left(\lambda(g)\right)$ for all $w \in \mathcal{W}(H)$. This can be proved using the fact that any Weyl transformation is a sequence of Weyl reflections w_j in the hyperplanes perpendicular to the simple coroots α_j^*. It is thus sufficient to prove that λ commutes with w_j for all unbroken simple roots. We have

$$\begin{aligned}\lambda\left(w_j(g)\right) = \lambda\left(g - \frac{2g \cdot \alpha_j^*}{\alpha_j^{*2}}\alpha_j^*\right) &= \lambda(g) - \frac{2g \cdot \alpha_j^*}{\alpha_j^{*2}}\lambda(\alpha_j^*) \\ = \lambda(g) - \frac{2\lambda(g) \cdot \alpha_j^*}{\alpha_j^{*2}}\alpha_j^* &= w_j\left(\lambda(g)\right).\end{aligned} \tag{2.67}$$

Note that for the unbroken roots $\lambda(\alpha_j) = \alpha_j$ and that λ is an isometry as discussed in section 2.3.3 and thus leaves the inner product invariant.

Secondly for the proof of the proposition we use the fact that for a lowest weight $\tilde{h}$ we have $w(\tilde{h}) = \tilde{h} + n_j\alpha_j^*$ with $n_j \geq 0$ for any $w \in \mathcal{W}(H^*)$, see for example chapter 10 to 13 of [47]. For $\tilde{g}$ and any $w \in \mathcal{W}(H^*) = \mathcal{W}(H)$ we now get:

$$\begin{aligned}\lambda\left(w(\tilde{g})\right) = w\left(\lambda(\tilde{g})\right) &= w(\tilde{h}) \\ &= \tilde{h} + n_j\alpha_j^* = \lambda(\tilde{g}) + n_j\lambda(\alpha_j^*) \\ &= \lambda(\tilde{g} + n_j\alpha_j^*).\end{aligned} \tag{2.68}$$

Consequently in terms of the unbroken simple coroots of G we find $w(\tilde{g}) = \tilde{g} + n_j\alpha_j^*$ where $n_j \geq 0$. Thus for the all fundamental weights of G we have $2\lambda_i \cdot w(\tilde{g}) \geq 0$ if $2\lambda_i \cdot \tilde{g} \geq 0$. □

Note that the set of positive reduced magnetic charges is a subset of the Murray cone and can be obtained by modding out the residual Weyl group. The set of Weyl orbits in the Murray cone is a physically important object; it corresponds to the magnetic charge sectors of the theory. This follows from the fact that a magnetic charge g is defined only modulo the action of the residual Weyl group. For this reason we shall introduce a set called the fundamental Murray cone which is bijective to the set of of Weyl orbits in the Murray cone. The set of positive reduced magnetic charges can of course be identified with the fundamental Murray cone. However, it would be more appropriate to call this set the anti-fundamental Murray cone. We recall that a reduced magnetic charge $\tilde{g}$ satisfies $\alpha_j \cdot \tilde{g} \leq 0$ for all unbroken simple roots α_j. It follows from this condition that $\tilde{g}$ can be identified with a lowest weight of H^*. Similarly, we can define the subset of the Murray cone $\{g : \alpha_j \cdot \tilde{g} \geq 0\}$. These magnetic charges now map to the fundamental Weyl chamber of H^*, hence we call this set the fundamental Murray cone. We thus find that the magnetic charge sectors are labelled by dominant integral weights of the residual gauge group. A similar conclusion was drawn for singular monopoles by Kapustin [48].

CHAPTER 3

FUSION RULES FOR SMOOTH BPS MONOPOLES

The magnetic charges carried by smooth BPS monopoles in Yang-Mills-Higgs theory with arbitrary gauge group G spontaneously broken to a subgroup H are restricted by a generalised Dirac quantization condition and by an inequality due to Murray. These conditions have been discussed in chapter 2. Geometrically, the set of allowed charges is a solid cone in the coroot lattice of G, which we call the Murray cone. As argued in section 2.3.4 magnetic charge sectors correspond to points in the Murray cone divided by the Weyl group of H.

The goal of this chapter is to determine which charge sectors contain indecomposable monopoles and which contain composite monopoles, and to find the rules according to which charge sectors are composed or "fused". Our success in finding a consistent set of rules provides an a posteriori justification of the definition of charge sectors. We begin, in section 3.1, by determining the additive structure of the Murray cone and the fundamental Murray cone. In both cases this results in a unique set of indecomposable charges which generate the cone. For Dirac monopoles similar sets of generating charges are introduced. We show that the generators of the fundamental Murray cone generate a subring in the representation ring of the residual gauge group. In the appendix A we construct an algebraic object whose representation ring is identical to to this special subring.

In order to support the interpretation of the indecomposable magnetic charges as building blocks of decomposable charges we review basic facts about moduli spaces of BPS monopoles in section 3.2. By analyzing the dimensions of these spaces we show that the decomposable charges for smooth BPS monopoles correspond to multi-monopole configurations built up from basic monopoles associated to the generating charges *provided*

we work within the fundamental Murray cone. The additive structure of the fundamental Murray cone thus provides candidate fusion rules for the magnetic charge sectors.
We find further support for these classical fusion rules in section 3.3 by drawing on existing results regarding the patching of monopoles, in particularly in the work of Dancer on BPS monopoles in an $SU(3)$ theory broken to $U(2)$. We briefly discuss similar results for singular BPS monopoles obtained by Kapustin and Witten [18] and speculate on the implications for the semi-classical fusion rules.

3.1 GENERATING CHARGES

As we have seen in section 2.3 consistency conditions on the charges of magnetic monopoles give rise to certain discrete sets of magnetic charges. In the case of singular monopoles this set is nothing but the weight lattice of the dual group H^*. The set of charges of smooth monopoles in a theory with adjoint symmetry breaking corresponds to the root lattice of the dual group G^*. Alternatively one can view this set as a subset in the weight lattice of the residual dual gauge group $H^* \subset G^*$. In the BPS limit the minimal energy configurations satisfy an even stronger condition which gives rise to the so-called Murray cone in the root lattice of G^*. Both the weight lattice of H^* and the Murray cone in the root lattice of G^* contain an important subset which is obtained by modding out the Weyl group of H^*. For singular monopoles one simply obtains the set of dominant integral weights, i.e. the fundamental Weyl chamber of H^*. In the case of smooth BPS monopoles modding out the residual Weyl group is equivalent to restricting the charges to the fundamental Murray cone.

In each case we want to find a set of minimal charges that generate all remaining charges via positive integer linear combinations. As it turns out this problem is most easily solved for the Murray cone. In the latter case the generators can be identified as the coroots with minimal topological charges. Below we shall prove this for any compact, connected semi-simple Lie group G and arbitrary symmetry breaking. For the weight lattice $\Lambda(H^*)$ one can give a generic description for a small set of generators. To find a smallest set of generators one needs to know some detailed properties of H^*. The generators the fundamental Weyl chamber and the fundamental Murray cone are not easily identified in general either. In all these cases we shall therefore restrict ourselves to some clear examples.

The physical interpretation of the generating charges is that the monopoles with these minimal charges are the building blocks of all monopoles in the theory. We shall therefore call monopoles with minimal charges in the weight lattice of H^* or in the Murray cone in G^* *fundamental monopoles*. The monopoles corresponding to the generators of the fundamental Weyl chamber and those related to the fundamental Murray cone both are called *basic monopoles*. In section 3.2 and 3.3 we study to what extent these notions make sense in the classical theory.

3.1.1 Generators of the Murray cone

Given two allowed magnetic charges g and g', that is two magnetic charges satisfying the Dirac condition (2.40) and the Murray condition (2.65), one can easily show that the linear combination $ng + n'g'$ with $n, n' \in \mathbb{N}$ again is an allowed magnetic charge. This raises the question whether all allowed magnetic charges can be generated from a certain minimal set of charges. This would mean that all charges can be decomposed as linear combinations of these generating charges with positive integer coefficients. The minimal set of generating charges is precisely the set of indecomposable charges. These indecomposable charges cannot be expressed as a non-trivial positive linear combination of charges in the Murray cone. It is obvious that such a set exists. It is also not difficult to show that such a set is unique. This follows from the fact all negative magnetic charges are excluded by the Murray condition. Despite its existence and uniqueness we do not know a priori what the set of generating charges is, let alone that we can be sure it is reasonably small or even finite.

There are some charges which are certainly part of the generating set, namely those for which the corresponding topological charges are minimal. These are the allowed charges g such that $2\lambda_i \cdot g = 1$ for one particular broken fundamental weight λ_i and $2\lambda_j \cdot g = 0$ for all other broken fundamental weights.

Proposition 3.1 *Topologically minimal charges are indecomposable.*

Proof. If an allowed charge g with a minimal topological component can be decomposed into two allowed charges, $g = g' + g''$ then one of these, say g', would have a topological component equal to zero. This means that $2\lambda_i \cdot g' = 0$ for all broken fundamental weights λ_i, implying that $g' = \sum_i m_i \alpha_i^*$ with only unbroken roots α_i and $m_i \geq 0$. If $\{\alpha_i\}$ is a set of simple roots of $H \subset G$ then so is $\{\alpha'_i\}$ with $\alpha'_i = -\alpha_i$. Since the Weyl group acts transitively on the bases of simple roots there exists an element in $\mathcal{W}(H)$ that takes all unbroken roots α_i to $\alpha'_i = -\alpha_i$. With respect to the basis (α'_i) we have $g' = \sum_i m'_i \alpha'_i$ with $m'_i \leq 0$. This implies that g' only satisfies the Murray condition if $g' = 0$ showing that g is indecomposable. □

We now wish to identify these topologically minimal charges. As a first step we shall show that some of the coroots, that is roots of G^* are contained in the set of topologically minimal charges. Note that there always exist coroots with topologically minimal charges, these correspond to the broken simple roots. If the residual symmetry group is non-abelian the set of topological minimal coroots is larger than the set of broken simple roots. In any case the whole set of topologically minimal coroots lies in the Murray cone.

Proposition 3.2 *Any coroot α^* with $2\lambda_j \cdot \alpha^* = 1$ for one of the broken fundamental weights and which is orthogonal to the other broken fundamental weights, satisfies the Murray condition.*

Proof. We shall first show that $\alpha^* \cdot \mu \geq 0$. As argued in section 2.3.4 we take μ to lie in the closure of the fundamental Weyl chamber, i.e. $\mu = 2\sum_i \mu_i \lambda_i$ with $\mu_i \geq 0$. Thus $\alpha^* \cdot \mu = \mu_j \geq 0$. If $\alpha^* \cdot \mu = 0$, α would be an unbroken root and as such orthogonal to all broken fundamental weights. This is clearly not the case since $2\lambda_j \cdot \alpha^* = 1$. We conclude that $\alpha^* \cdot \mu > 0$ and hence that α^* is a positive coroot.
It is now easy to show that α^* does indeed satisfy Murray's condition. Since the Weyl group is the symmetry group of the (co)root system we have for any $w \in \mathcal{W}(H) \subset \mathcal{W}(G)$, that $w(\alpha^*)$ is another coroot. Moreover $w(\alpha^*)$ is positive since the residual Weyl group leaves the Higgs VEV invariant: $w(\alpha^*) \cdot \mu = \alpha^* \cdot w^{-1}(\mu) = \alpha^* \cdot \mu$. We thus have that $w(\alpha^*) \cdot \mu > 0$ for any $w \in \mathcal{W}(H)$. Equaling some root of G^* the positivity of $w(\alpha^*)$ implies that it can be expanded in simple positive coroots with all coefficient greater than zero: $2\lambda_j \cdot w(\alpha^*) \geq 0$. We finally find that $2w(\lambda_i) \cdot \alpha^* \geq 0$ for all fundamental weights and for all elements in the residual Weyl group. □

It was easily shown that topologically minimal charges satisfying the Murray condi-

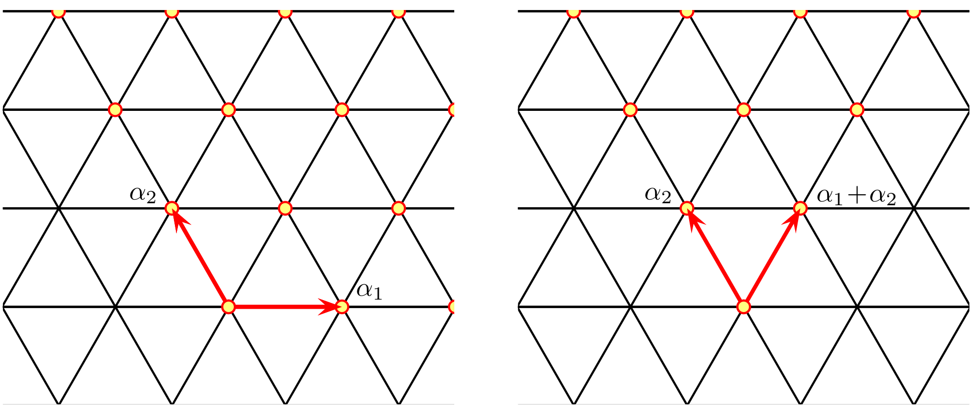


Figure 3.1: *In the picture above the generators of the Murray cones of $SU(3)$ are depicted. If the gauge group is maximally broken (left) to $U(1) \times U(1)$, the generators correspond to the simple roots of SU(3). Both generating charges have distinct unit topological charges. For minimal symmetry breaking (right) where the gauge group is $U(2)$, the Murray cone is further restricted by the Murray condition. The generating magnetic charges do have distinct holomorphic charges related by the Weyl group. Their topological charges both equal 1.*

tion are indecomposable charges within the Murray cone. Furthermore we have seen that these topologically minimal charges contain the set of coroots with topologically minimal charges. We will now prove that these coroots do not only constitute the complete set of minimal topological charges in the Murray cone, they actually form the full set of indecomposable charges. For $G = SU(3)$ these facts are easily verified in figure 3.1 where the Murray cones and its generators are drawn for the two possible patterns of adjoint symmetry breaking. Below we prove that the minimal topological charges generate the full Murray cone. Consequently the set of minimal topological charges must coincide with the complete set of indecomposable charges.

Proposition 3.3 *The coroots with minimal topological charges generate the Murray cone*

Proof. The outline of the proof is as follows. We slice up the Murray cone according to the topological charges in such a way that each layer corresponds to a unique representation of the dual residual group. For unit topological charges we show that the weights correspond to the coroots with unit topological charges. Finally we show that the representations for higher topological charges pop up in the symmetric tensor products of representations with unit topological charges.

Consider $G \to H$ where H is locally of the form $U(1)^s \times K$. We split the r roots of the gauge group G into s broken roots (α_i) with $0 < i \leq s$ and $r - s$ unbroken roots (α_j) with $s < j \leq r$. The magnetic charges are thus expanded as $g = \sum_i m_i \alpha_i^* + \sum_j h_j \alpha_j^*$. Without loss of generality we can assume G to be simply connected just like in the proof of proposition 2.1. In that same proof we also defined an isomorphism λ from the coroot lattice $\Lambda^*(G)$ to the weight lattice $\Lambda(H^*)$ of H^*. Since H^* is locally of the form $U(1)^l \times K^*$ with K semi-simple, $\lambda(g)$ can be expressed in terms of the $U(1)$ charges and a weight of K^*. While the abelian charges are identified with the topological charges m_i, the Dynkin labels of the non-abelian charge are by $k_j = 2\alpha_j^* \cdot g / {\alpha_j^*}^2$. Being sums of multiples of the entries of the Cartan matrix of G^* these labels are indeed integers. Moreover for vanishing holomorphic charges only the off-diagonal entries contribute so that $k_j \leq 0$. Consequently for any $g \in \Lambda^*(G)$ we have:

$$
\begin{aligned}
\lambda(g) &= \lambda\left(m_i \alpha_i^* + h_j \alpha_j^*\right) \\
&= \lambda\left(m_i \alpha_i^*\right) + \lambda\left(h_j \alpha_j^*\right) \\
&= h_-(m_i) + h_j \alpha_j^*.
\end{aligned}
\tag{3.1}
$$

where $h_-(m_i)$ is a lowest weight that only depends on the topological charges. We shall prove that for a fixed set of positive topological charges $\{m_i\}$ the magnetic charges in the Murray cone are in one-to-one relation with the weights of the irreducible representation of H^* labelled by $h_-(m_i)$. To show this we use two important facts. First a weight h is in the representation defined by h_- if and only if for the lowest weight $\tilde{h}$ in the Weyl orbit of h one has $\tilde{h} = h_- + n_j \alpha_j^*$ where $n_j \geq 0$. Second, the map λ commutes with the residual Weyl group.

First we shall show that for a magnetic charge g in the Murray cone $\lambda(g)$ is a weight

in the $h_-(m_i)$ representation. As a superficial consistency check we note that $\lambda(g)$ and $h_-(m_i)$ differ by an integer number of roots of H^* given by the holomorphic charges. The lowest weight in the Weyl orbit of $\lambda(g)$ is given by the image of the reduced magnetic charge $\lambda(\tilde{g})$, as explained in the proof of proposition 2.1. It follows from the Murray condition (2.65) that $\tilde{g}$ is of the form $\tilde{g} = m_i\alpha_i^* + h_j''\alpha_j^*$ where $h_j'' \geq 0$. Consequently $\lambda(\tilde{g}) = h_-(m_i) + n_j\alpha_j^*$ where $n_j \geq 0$.
To prove the converse we take a weight h in the representation defined $h_-(m_i)$ with $m_i \geq 0$. We need to prove that g with $\lambda(g) = h$ satisfies the Murray condition. This is done as follows. The triple $(h_-(m_i), \tilde{h}, h)$ of weights in $\Lambda(H^*)$ can be mapped to a triple $(g_-(m_i), \tilde{g}, g)$ of elements in the coroot lattice $\Lambda^*(G)$ by the inverse of λ. Next we show that $g_-(m_i), \tilde{g}$ and g satisfy the Murray condition. We have $g_-(m_i) = m_i\alpha_i^*$ so that $\lambda(g_-) = \lambda_-(m_i)$ and $m_i \geq 0$. The broken simple coroots satisfy the Murray condition and hence $g_-(m_i)$ lies in the Murray cone. $\tilde{g}$ is given by $\tilde{g} = g_-(m_i) + n_j\alpha_j^*$ so that $\lambda(\tilde{g}) = \lambda(g_-(m_i)) + n_j\alpha_j^* = \tilde{h}$. Since $\tilde{g}$ maps to the anti-fundamental Weyl chamber of H^* and has a positive expansion in simple coroots it satisfies the Murray conditions as follows from proposition 2.1. Finally since λ respects the residual Weyl group and $\tilde{h}$ is in the Weyl orbit of h we find that g is in the Weyl orbit of $\tilde{g}$. With $\tilde{g}$ satisfying the Murray condition it is easy to show that g also obeys the condition.
The coroots of G form the nonzero weights of the adjoint representation of G^*. Under symmetry breaking the adjoint representation maps to a reducible representation of H^*. We are particularly interested in the irreducible factors corresponding to unit topological charges. Coroots with unit topological charge, i.e. $m_i = \delta_{ik}$, equal a broken simple coroot α_k^* up to unbroken roots. We have seen in proposition 3.2 that coroots with unit topological charge satisfy the Murray condition. Hence the previous discussion tells us that such coroots are mapped to the weight space of the representation labelled by $\lambda(\alpha_k^*)$. The weight $\lambda(\alpha_k^*)$ itself corresponds to $g = \alpha_k^*$. We now see that each weight in the $\lambda(\alpha_k^*)$-representation must not only correspond to a magnetic charge in the coroot lattice of G but in fact to a coroot, otherwise the coroot system would not constitute a proper representation of H^*.
We can now finish the proof. Each element in the Murray cone is the weight in a representation labelled by $h_-(m_i)$. Such representations only depend on the topological charges. Moreover the lowest weights are additive with respect to the topological charges: $h_-(m_i) + h_-(m_i') = h_-(m_i + m_i')$. Consequently every such lowest weight is of the form $\sum_i m_i\lambda(\alpha_i^*)$. The representation labelled by $h_-(m_i)$ is obtained by the symmetric tensor product of representations labelled by $\lambda(\alpha_i)$. A weight in the product representation equals a sum of weights from the $\lambda(\alpha_i^*)$ representations. By identifying the weights with magnetic charges we find that all charges is the Murray cone equal a sum of coroots with unit topological charges. □

3.1.2 GENERATORS OF THE MAGNETIC WEIGHT LATTICE

In this section we want to describe the generators of the magnetic charge lattice for singular monopoles in a theory with gauge group H. This charge lattice can be identified with the weight lattice $\Lambda(H^*)$ of the dual group H^* as discussed in section 2.3.1. As for the Murray cone it is obvious that a minimal set of generating charges exists such that all charges are linear combinations of these generating charges with positive integer coefficients. The difference with the Murray cone, however, is that the generating set is not necessarily unique. We shall give some simple examples below to illustrate this, but we already note that the underlying reason for this is that the weight lattice of H^* is closed under inversion.

Using some textbook results on Lie group theory is easy to find a relatively small set of generators: let V be a faithful representation of H^* and V^* its conjugate representation. Any irreducible representation of H^* is contained in the tensor products of V and V^*, see e.g section VIII of [49] for a proof. Since the weights of $V_1 \otimes V_2$ are given by the sums of the weights of V_1 and V_2 we now find that any weight of an irreducible representation of H^* is a linear combination of weights of V and V^* with positive coefficients. Since any weight in $\Lambda(H^*)$ is contained in an irreducible representation of H^* we have found that the weights of V and V^* generate the magnetic weight lattice. Note that if this faithful representation V is self-conjugate the weight lattice is obviously generated by the non-zero weights of V. This happens for example for $SO(n)$ and $Sp(2n)$ which have only self-conjugate representations. To find a small set of generators one should take the non-zero weights of a smallest faithful representation and its conjugate representation, i.e. the fundamental representation and its conjugate representation.
The recipe above does not necessarily give a smallest set of generators since there still might be some double counting. We mention two examples. First V^* might be contained in the tensor products of V. This happens for example for $SU(n)$: the representation $\bar{\mathbf{n}}$ is given by the $(n-1)$th anti-symmetric product of $\mathbf{n}$. Second some weights of V may be decomposable within V. Consider for example $SU(n)/\mathbb{Z}_n$. The weight lattice of this group corresponds to the root lattice of $SU(n)$ and for V one can take the adjoint representation whose weights are the roots of $SU(n)$. Note that all roots can be expressed as positive linear combinations of the simple roots and their inverses in the root lattice.
When H^* is a product of groups the defining representation is reducible and falls apart into irreducible components. Each of these irreducible representations has trivial weights for all but one of the group factors. This agrees with the fact that in this case the weight lattice of H^* is a product of weight lattices.
In table 3.1 we give the representation or representations whose nonzero weights constitute a minimal generating set of the magnetic weight lattice $\Lambda(H^*)$. The corresponding electric groups H were mentioned in tables 2.1 and 2.2.

H^*	$\{V\}$
$SU(n)$	$\{\mathbf{n}\}$
$Sp(2n)$	$\{\mathbf{2n}\}$
$SO(n)$	$\{\mathbf{n}\}$
$(U(1)\times SU(n))/\mathbb{Z}_n$	$\{\mathbf{n}_1, \bar{\mathbf{n}}_{-1}\}$
$U(1)\times SO(2n+1)$	$\{(\mathbf{2n+1})_0, \mathbf{1}_1, \mathbf{1}_{-1}\}$
$(U(1)\times Sp(2n))/\mathbb{Z}_2$	$\{\mathbf{2n}_1, \mathbf{2n}_{-1}\}$
$(U(1)\times SO(2n))/\mathbb{Z}_2$	$\{\mathbf{2n}_1, \mathbf{2n}_{-1}\}$

Table 3.1: *Generators of the magnetic weights lattice $\Lambda(H^*)$ in terms of representations of the dual group H^*. The boldface numbers give the dimensionality of the irreducible representations of the corresponding simple Lie groups, their conjugate representations are distinguished by an extra bar. The subscripts denote $U(1)$-charges.*

3.1.3 Generators of the fundamental Weyl chamber

The charges of singular monopoles in a theory with gauge group H take values in the weight lattice of the dual group H^*. This weight lattice has a natural subset: the weights in the fundamental Weyl chamber. If H is semi-simple and has trivial center H^* is semi-simple and is simply connected. In this particular case the generators of the fundamental Weyl chamber of H^* are immediately identified as the fundamental weights. If H^* is not simply connected or even not semi-simple the generating weights in the fundamental Weyl chamber are not that easily identified. The generating charges are, however, closely related to the generators of the representation ring, which are computed in chapter 23 of [50]. We shall explain this relation for the semi-simple, simply connected Lie groups. Finally we use the obtained intuition to compute the generators of the fundamental Weyl chamber for the dual groups in table 2.2 which occur in minimal symmetry breaking of classical groups. In the next section we shall use similar methods to find the generators of the fundamental Murray cone.

The representation ring $R(H^*)$ is the free abelian group on the isomorphism classes of irreducible representations of H^*. In this group one can formally add and subtract representations. The tensor product makes $R(H^*)$ into a ring. We shall for now assume H^* to be a simple and simply connected Lie group of rank r so that its weight lattice Λ is generated by the r fundamental weights $\{\lambda_i\}$.
$R(H^*)$ is isomorphic to a certain ring of Weyl-invariant polynomials. We will review the proof following [50]. We shall start by introducing $\mathbb{Z}[\Lambda]$, the integral ring on Λ. By this we mean that any element in $\mathbb{Z}[\Lambda]$ can be written as $\sum_\Lambda n_\lambda e_\lambda$ where $n_\lambda \in \mathbb{Z}$ and $n_\lambda \neq 0$ for a finite set of weights. We thus see that e_λ is the basis element in $\mathbb{Z}[\Lambda]$ corresponding

to λ. The product in $\mathbb{Z}[\Lambda]$ is defined by $e_\lambda e_{\lambda'} = e_{\lambda+\lambda'}$. We thus see that $\mathbb{Z}[\Lambda]$ is nothing but a group ring on the abelian group Λ. Note that the additive and multiplicative unit are given by 0 and e_0 while the additive and multiplicative inverses of e_λ are given by respectively $-e_\lambda$ and $e_{-\lambda}$.

There is a homomorphism, denoted by Char, from the representation ring into $\mathbb{Z}[\Lambda]$ This map sends a representation V to $\text{Char}(V) = \sum \dim(V_\lambda)\, e_\lambda$, where $\dim(V_\lambda)$ equals the multiplicity with which the weight λ occurs in the representation V. It is easy to see that this map does indeed respect the ring structure.
The Weyl group $\mathcal{W}$ of H^* acts linearly on $\mathbb{Z}[\Lambda]$ and the action is defined by $w \in \mathcal{W}$: $e_\lambda \mapsto e_{w(\lambda)}$. To show that the action of $\mathcal{W}$ respects the multiplication in $\mathbb{Z}[\Lambda]$ one simply uses the fact that $\mathcal{W}$ acts linearly on Λ.
$\mathbb{Z}[\Lambda]$ contains a subring $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ consisting of elements invariant under the Weyl group. The claim is that $R(H^*)$ is isomorphic to $\mathbb{Z}[\Lambda]^{\mathcal{W}}$. It is easy to show that the image of Char is contained in $\mathbb{Z}[\Lambda]^{\mathcal{W}}$. Below we shall also prove surjectivity by using the fact that there is a basis of $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ that is generated out of certain representation of H^*. In the end we are of course interested in these generators.

To each dominant integral weight $\lambda \in \Lambda$ we associate an element $P_\lambda \in \mathbb{Z}[\Lambda]^{\mathcal{W}}$ by choosing $P_\lambda = \sum n_{\lambda'} e_{\lambda'}$ with $n_{w(\lambda')} = n_{\lambda'}$ for all $w \in \mathcal{W}$ and with $n_\lambda = 1$. For simplicity we take P_λ so that $n_{\lambda'} = 0$ if $\lambda - \lambda'$ is not a linear combination of roots. We now restrict the choice of P_λ so that for any dominant integral weight $\lambda' > \lambda$, $n_{\lambda'}$ vanishes. Note that λ is the highest weight of P_λ. One can now prove by induction that any set $\{P_\lambda\}$ satisfying the conditions above forms an additive basis for $\mathbb{Z}[\Lambda]^{\mathcal{W}}$.
We shall now make a rather special choice for the basis $\{P_\lambda\}$. For the fundamental weights λ_i we take P_{λ_i} to be $P_i = \text{Char}(V_i)$ were V_i is the irreducible representation of H^* with highest weight λ_i. For any other dominant integral weight $\lambda = \sum m_i \lambda_i$ we take $P_\lambda = \text{Char}\,(\otimes_i V_i^{m_i}) = \Pi_i\, P_i^{m_i}$. Since $\{P_\lambda\}$ is a basis for $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ any element in this ring can thus be written as a polynomial in the variables P_i with positive integer coefficients:

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[P_1, \ldots, P_r]. \tag{3.2}$$

As promised we have proven that $R(H^*)$ is isomorphic to $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ for H^* semi-simple and simply connected. In addition we have found that the generators of $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ correspond precisely to the generators of the fundamental Weyl chamber via the map $\lambda_i \mapsto P_i$. This is not very surprising because it was input for the proof of the isomorphism. So the interesting question is if we can really retrieve the generators of the fundamental Weyl chamber from $R(H^*)$. This can indeed be done by identifying the generators of Λ with the generators of $Z[\Lambda]$. We shall explain this below for $SU(n)$. Before we do so we want to make an important remark.
In the proof we used the fact that there is a basis P_λ where each P_λ can be identified with

Char(V_λ) and where V_λ is some representation with highest weight λ. Such a choice of basis always exist since one can take V_λ be the irreducible representation with highest weight λ. The fact that there is a generating set for the fundamental Weyl chamber is thus not crucial in the proof of the isomorphism between $R(H^*)$ and $\mathbb{Z}[\Lambda]^{\mathcal{W}}$.

We return to $\mathbb{Z}[\Lambda]$, where Λ is the weight lattice of $SU(n)$. As discussed in the previous section the weight lattice of $SU(n)$ is generated by the weights of the n-dimensional fundamental representation. Let us denote these weights by L_i and define

$$x_i = e_{L_i} \in \mathbb{Z}[\Lambda]. \tag{3.3}$$

Note that the vectors L_i are not linearly independent since $\sum_i L_i = 0$. We thus have

$$x_1 x_2 \cdots x_n = 1, \tag{3.4}$$

where $1 = e_0$ is the multiplicative unit of $Z[\Lambda]$. We find that any element e_λ can be written as monomial $\Pi_i\, x_i^{m_i}$ with positive coefficients m_i. Such monomials are unique up to factors $x_1 \cdots x_n$. Since $\{e_\lambda : \lambda \in \Lambda\}$ forms a basis for $\mathbb{Z}[\Lambda]$ we find:

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, \ldots x_n]/(x_1 \cdots x_n - 1). \tag{3.5}$$

The Weyl group of $SU(n)$ is the permutation group $\mathcal{S}_n$ and obviously permutes the indices of the x_is, see also appendix B.1. Consequently

$$R(SU(n)) = \mathbb{Z}[\Lambda]^{\mathcal{S}_n} = \mathbb{Z}[x_1, \ldots x_n]^{\mathcal{S}^n}/(x_1 \cdots x_n - 1). \tag{3.6}$$

To find the generators of $R(SU(n))$ we use the well known fact that any symmetric polynomial in n variables can be expressed as a polynomial of $a_k : k = 1, \ldots, n$ where a_k is the kth elementary symmetric function of x_i given by:

$$a_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}. \tag{3.7}$$

Note that $a_n = x_1 \cdots x_n$ is identified with 1 in $R(SU(n))$. We have thus established the isomorphism:

$$R(SU(n)) = \mathbb{Z}[a_1, \ldots, a_{n-1}]. \tag{3.8}$$

Our conclusion is that the first $n-1$ elementary symmetric functions form a minimal set generating the representation ring of $SU(n)$. It should not be very surprising that for $i < n$ $a_i = P_i =$ Char(V_i) where V_i is the irreducible representation with highest weight λ_i. It is nice to note that $V_i = \wedge^i V$ where V is the fundamental representation of $SU(n)$ and that $\wedge^n V = 1$ the trivial representation.

For $SO(2n+1)$, $Sp(2n)$, and $SO(2n)$ the fundamental representation has $2n$ nonzero weights $\pm L_i : i = 1, \ldots n$. By identifying $x_i^{\pm 1} = e_{\pm L_i}$ one finds that the group ring on the weight lattice is isomorphic to $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. As shown in [50] the representation rings are given by polynomial rings of the form:

$$R(SO(2n+1)) = \mathbb{Z}[b_1, \cdots, b_n] \tag{3.9}$$
$$R(Sp(2n)) = \mathbb{Z}[c_1, \cdots, c_n] \tag{3.10}$$
$$R(SO(2n)) = \mathbb{Z}[d_1, \cdots, d_{n-1}, d_n^+, d_n^-]. \tag{3.11}$$

The polynomials b_k, c_k and d_k can all be chosen to equal the elementary symmetric functions in the $2n$ variables $\{x_i^{\pm}\}$. The polynomials $d_n^{\pm}$ can be expressed as $(d^{\pm})^2$. d^+ and d^- correspond to the two spinor representations of $SO(2n)$:

$$d^{\pm} = \mathrm{Char}(S_{\pm}) = \sum_{s_1 \cdots s_n = \pm 1} \sqrt{x_1^{s_1} \cdots x_n^{s_n}}. \tag{3.12}$$

It is easy to check that $d_n^{\pm}$ are indeed polynomials.
To explain why $R(SO(2n))$ has an extra generator compared to the other groups we note that its Weyl group is given by $\mathcal{S}_n \ltimes \mathbb{Z}_2^{n-1}$ whereas the Weyl groups of $SO(2n+1)$ and $Sp(2n)$ are given by $\mathcal{S}_n \ltimes \mathbb{Z}_2^n$. This means that the Weyl groups act on the non-zero weights of the fundamental representations by permuting the indices and changing the signs of the weights, but for $SO(2n)$ only an even number of sign changes is allowed, see also appendix B.1. Consequently the generators of $R(SO(2n))$ do not have to be invariant under for example of $x_1 \mapsto x_1^-$ and hence the generator d_n can be decomposed into d_n^+ and d_n^-.
for completeness we mention that the highest weights of b_k, c_k and d_k are given by the highest weights of the anti-symmetric tensor products $\wedge^k V$ of the corresponding fundamental representation V. The highest weights of $d_n^{\pm}$ are given by twice the highest weight of the spinor representations $S^{\pm}$.

We finally want to identify the generators of the fundamental Weyl chamber for some groups that arise in minimal symmetry breaking of classical groups. As discussed in section 3.1.2 the weight lattice Λ of $U(n)$ is generated by the weights of its n-dimensional representation $\mathbf{n}_1$ and those of its conjugate representation $\bar{\mathbf{n}}_{-1}$. Let us denote the weights of $\mathbf{n}_1$ by $\{L_i\}$ and define $x_i = e_{L_i} \in \mathbb{Z}[\Lambda]$. The weights of $\bar{\mathbf{n}}_{-1}$ are given by $\{-L_i\}$. We thus immediately find the following isomorphism for the group ring on the weight lattice of $U(n)$:

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[x_1, x_{-1}, \ldots, x_n, x_n^{-1}]. \tag{3.13}$$

To find the generators of the representation ring $R(U(n)) = \mathbb{Z}[\Lambda]^{\mathcal{W}}$ we note that the Weyl group $\mathcal{W} = \mathcal{S}_n$ of $U(n)$ permutes the indices of the generators of $\mathbb{Z}[\Lambda]$ but does not change any of the signs as happened for the classical groups discussed right above. This implies that $R(U(n))$ is generated by $\{a_k : k = 1, \ldots, n\}$ the elementary symmetric polynomials in x_i and $\{\bar{a}_k : k = 1, \ldots, n\}$ the elementary symmetric polynomials in the

variables $\{x_i^{-1}\}$. Note that $a_n = x_1 \cdots x_n$ is invertible in the representation ring and its inverse is given by $\bar{a}_n = (x_1 \cdots x_n)^{-1}$.
The generators we have found for $R(U(n))$ are not completely independent since:

$$a_k a_n^{-1} = \sum_{i_{j-1}<i_j<i_{j+1}} x_{i_1} \cdots x_{i_k} (x_1 \cdots x_n)^{-1} = \sum_{i_{j-1}<i_j<i_{j+1}} (x_{i_1} \cdots x_{i_{n-k}})^{-1} = \bar{a}_{n-k}. \quad (3.14)$$

The representation ring of $U(n)$ can thus be identified with the polynomial ring:

$$R(U(n)) = \mathbb{Z}[a_1, \ldots, a_n, a_n^{-1}]. \quad (3.15)$$

The generating polynomials a_k and a_n^{-1} are indecomposable in the representation ring, their highest weights thus form a minimal set generating the fundamental Weyl chamber of $U(n)$. We finally mention that $a_k = \text{Char}(\wedge^k V)$, where V is the fundamental representation of $U(n)$. Moreover $\wedge^n V$ is the one dimensional representation that acts by multiplication with det(g) where $g \in (U(n))$ This representation is invertible and $a_n^{-1} = \text{Char}\left((\wedge^n V)^{-1}\right)$.

Since $U(1)\times SO(2n+1)$ is a product of groups its representation ring is simply $R(U(1))\times R(SO(2n+1)$. The representation ring of $U(1)$ can be identified with the polynomial ring $\mathbb{Z}[x_0, x_0^{-1}]$ where $x_0^{\pm 1} = \text{Char}(V^{\pm 1})$ and V the fundamental representation of $U(1)$. There is, however, an alternative description of the representation ring which will prove to be valuable in the next section. Let $\{L_0, L_1, L_1', \ldots, L_n, L_n'\}$ be the weights of the fundamental representation of $U(1) \times SO(2n+1)$, i.e. the representation with unit $U(1)$ charge. Define $\{x_0, x_1, x'_1, \ldots x_n, x_n'\}$ to be the images of these weights in the group ring $\mathbb{Z}[\Lambda]$ of the weight lattice. It is not too hard to show that $\mathbb{Z}[\Lambda]$ is isomorphic to $\mathbb{Z}[x_0, x_0^{-1}, x_1, x'_1, \ldots x_n, x_n']/I$ where I is the ideal generated by the relations $x_i x'_i = x_0^2$. Moreover one can prove that

$$R(U(1) \times SO(2n+1)) = \mathbb{Z}[x_0, x_0^{-1}, b_1, \ldots, b_n], \quad (3.16)$$

where $b_k = \text{Char}(\wedge^k V)$ is the kth elementary symmetric polynomial in the $2n+1$ variables $x_0, x_1, x'_1, \ldots, x_n, x'_n$. The highest weights of these generating polynomials correspond to the minimal set of generating charges in the fundamental Weyl chamber.

By mapping the weights of the fundamental representation and its conjugate representations to $\mathbb{Z}[\Lambda]$ on finds that group rings on the weight lattices of $(U(1) \times Sp(2n))/\mathbb{Z}_2$ and $(U(1) \times SO(2n))/\mathbb{Z}_2$ can be identified with the polynomial ring

$$\mathbb{Z}[x_1, x_1^{-1}, x'_1, x'^{-1}_1, \ldots, x_n, x_n^{-1}, x_n', x'_{n-1}]/I, \quad (3.17)$$

where I is the ideal generated by the relations $x_i x'_i = x_j x'_j$. Note that these relations imply that $x_1 x_1'$ is invariant under the Weyl group that permutes the indices and swaps primed variables with their unprimed counterparts. One can now show that the representation rings $R\left((U(1) \times Sp(2n))/\mathbb{Z}_2\right)$ and $R\left((U(1) \times SO(2n))/\mathbb{Z}_2\right)$ can be identified as quotient rings of respectively:

$$\mathbb{Z}[x_1 {x'}_1, (x_1 {x'}_1)^{-1}, c_1, \ldots, c_n, \bar{c}_1, \ldots, \bar{c}_n] \tag{3.18}$$

and

$$\mathbb{Z}[x_1 {x'}_1, (x_1 {x'}_1)^{-1}, d_1, \ldots, d_{n-1}, d_n^+, d_n^-, \bar{d}_1, \ldots, \bar{d}_{n-1}, \bar{d}_n^+, \bar{d}_n^-], \tag{3.19}$$

where c_k and d_k are the elementary symmetric polynomials in the $2n$ variables $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$. The functions $\bar{c}_k$ and $\bar{d}_k$ are similar elementary symmetric polynomials expressed in terms of the inverted variables. Explicit expressions for $d_n^\pm = (d^\pm)^2$ can be found from formula (3.12) where the inverted variables should be replaced by the primed variables. Finally $\bar{d}_n^\pm$ is found by substitution of the inverted variables in $d_n^\pm$. The generating set of polynomials we have found is not the minimal set. This follows from the fact that $x_j^{-1} = (x_j x'_j)^{-1} x'_j = (x_1 x'_1)^{-1} x'_j$. Consequently one finds:

$$R\left((U(1) \times Sp(2n))/\mathbb{Z}_2\right) = \mathbb{Z}[x_1 {x'}_1, (x_1 {x'}_1)^{-1}, c_1, \ldots, c_n] \tag{3.20}$$

$$R\left((U(1) \times SO(2))/\mathbb{Z}_2\right) = \mathbb{Z}[x_1 {x'}_1, (x_1 {x'}_1)^{-1}, d_1, \ldots, d_{n-1}, d_n^+, d_n^-]. \tag{3.21}$$

The highest weights of these generating polynomials are the generators of the fundamental Weyl chamber of the two groups.

3.1.4 Generators of the fundamental Murray cone

The fundamental Murray cone, just like the Murray cone, contains a unique set of indecomposable charges. The uniqueness of this set is a consequence of the fact that the fundamental Murray cone does not allow for invertible elements. The main difference with the Murray cone, however, is that the generators for the fundamental Murray cone are not easily computed. After a general discussion we shall therefore only determine the generators for a couple of cases that correspond to minimal symmetry breaking of classical groups. The approach we use is closely related to the computation of the generators of the fundamental Weyl chamber as discussed in the previous section and can in principle be applied to any gauge group and for arbitrary symmetry breaking.

Note that this whole exercise only makes sense if the fundamental Murray cone is closed under addition. At the beginning of section 3.1.1 we argued that the Murray cone is closed under this operation by evaluating the defining equations. For the fundamental Murray cone similar considerations apply. For g to be in the fundamental Weyl chamber of the Murray cone we have the extra condition $g \cdot \alpha_i \geq 0$ for all unbroken roots α_i. It is now easily seen that if both g and g' satisfy this condition then $g + g'$ will satisfy it too, as will any linear combination of these charges with positive integer coefficients. This proves that the fundamental Murray is closed under addition of charges.

Instead of computing the generators of the fundamental Murray cone directly by evaluating the Murray condition we shall determine the indecomposable generators of a certain representation ring. We shall start by describing this ring. Let G be a compact, semi-simple group broken to H via an adjoint Higgs field. Without loss of generality we can assume G to be simply connected since this does not change the set of magnetic charges. Under this condition the magnetic weight lattice $\Lambda := \Lambda(H^*)$ is isomorphic to the root lattice of G^*. The ring we want to consider is the free abelian group on the irreducible representations of H^* with weights in the Murray cone. These irreducible representations of H^* are labelled by dominant integral weights in $\Lambda_+ \subset \Lambda$ and can be identified with the fundamental Murray cone as a set. Note that since the Murray cone is closed under addition this set of representations is closed under the tensor product. As we prove in the appendix there exists an algebraic object, but not a group, having a complete set of irreducible representations labelled by the magnetic charges in the Murray cone. Let us denote this object by H^*_+. The representation ring we are discussing here is thus precisely the representation ring $R(H^*_+)$.

Just as in the previous section we now introduce a second ring $\mathbb{Z}[\Lambda_+]$ that turns out to be quite useful. $\mathbb{Z}[\Lambda_+]$ has a basis $\{e_\lambda : \lambda \in \Lambda_+\}$. Since Λ_+ is closed under addition $\mathbb{Z}[\Lambda_+]$ is indeed closed under multiplication. The multiplicative identity is given by $1 = e_0$. The basis elements e_λ of $\mathbb{Z}[\Lambda_+]$ are not invertible under multiplication since $e_{-\lambda}$ is not contained in $\mathbb{Z}[\Lambda_+]$. Finally we introduce the ring $\mathbb{Z}[\Lambda_+]^{\mathcal{W}}$ consisting of the Weyl invariant elements in $\mathbb{Z}[\Lambda_+]$. Note that $\mathbb{Z}[\Lambda_+] \subset \mathbb{Z}[\Lambda]$ and $\mathbb{Z}[\Lambda_+]^{\mathcal{W}} \subset \mathbb{Z}[\Lambda]^{\mathcal{W}}$. By using arguments almost identical to arguments mentioned in the previous section one can show that $R(H^*_+)$ is isomorphic to $\mathbb{Z}[\Lambda_+]^{\mathcal{W}}$. This last ring can be identified with a polynomial ring. The highest weights of the indecomposable polynomials can be identified with the generators of the fundamental Murray cone.

We shall identify the generators of the fundamental Murray cone for the classical simply-connected groups $SU(n+1)$, $Sp(2n+2)$, $Spin(2n+3)$, and $Spin(2n+2)$ and for minimal symmetry breaking. The relevant residual electric groups and their magnetic dual groups are listed in table 2.2. One can show that the Murray cone in these cases is generated by the weights of the fundamental representation of H^* which are respectively n, $2n+1$, $2n$ and $2n$ dimensional.

Let us denote the weights of the fundamental representation of $U(n)$ by L_i where $i = 1, \ldots, n$. We define $x_i = e_{L_i}$. Since the weights L_i freely generate the Murray cone we immediately find

$$\mathbb{Z}[\Lambda_+] = \mathbb{Z}[x_1, \ldots, x_n]. \tag{3.22}$$

The Weyl group of $U(n)$ permutes the indices of the generators. Copying our results of the previous section we thus find the following isomorphism:

$$\mathbb{Z}[\Lambda_+]^{\mathcal{W}} = \mathbb{Z}[a_1, \ldots, a_n]. \tag{3.23}$$

where a_k are the elementary symmetric polynomials in the variables x_i. The highest weights of these indecomposable polynomials are the generators of the fundamental Murray cone for $SU(n+1)$ broken down to $U(n)$. Note that $\mathbb{Z}[\Lambda_+]^{\mathcal{W}}$ is obtained from $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ as given in formula (3.15) by removing the generator a_n^{-1}.

Let $\{L_0, L_1, L_1', \ldots, L_n, L_n'\}$ be the weights of the fundamental representation of $U(1) \times SO(2n+1)$. Define $\{x_0, x_1, x'_1, \ldots x_n, x_n'\}$ to be the images of these weights in the ring $\mathbb{Z}[\Lambda_+]$. $\mathbb{Z}[\Lambda_+]$ is isomorphic to

$$\mathbb{Z}[x_0, x_1, x'_1, \ldots x_n, x_n']/I, \tag{3.24}$$

where I is the ideal generated by the relations $x_i x'_i = x_0^2$. Moreover one can now prove that

$$\mathbb{Z}[\Lambda_+]^{\mathcal{W}} = \mathbb{Z}[x_0, b_1, \ldots, b_n], \tag{3.25}$$

where $b_k = \mathrm{Char}(\wedge^k V)$ is the kth elementary symmetric polynomial in the $2n+1$ variables $x_0, x_1, x'_1, \ldots, x_n, x'_n$. The highest weights of these generating polynomials correspond to the minimal set of generating charges in the fundamental Murray cone.

By mapping the weights of the fundamental representation to $\mathbb{Z}[\Lambda_+]$ for G equals $Sp(2n+2)$ or $SO(2n+2)$ the ring $\mathbb{Z}[\Lambda_+]$ can be identified with the polynomial ring

$$\mathbb{Z}[x_1, x'_1, \ldots, x_n, x_n']/I, \tag{3.26}$$

where I is the ideal generated by the relations $x_i x'_i = x_j x'_j$. One can now show that the representation rings can be identified as respectively:

$$\mathbb{Z}[x_1 x'_1, c_1, \ldots, c_n] \tag{3.27}$$

and

$$\mathbb{Z}[x_1 x'_1, d_1, \ldots, d_{n-1}, d_n^+, d_n^-], \tag{3.28}$$

where c_k and d_k are both elementary symmetric polynomials in the variables $x_1, \ldots, x_n$ and $x_1', \ldots, x_n'$. Explicit expressions for $d_n^{\pm} = (d^{\pm})^2$ are the same as the corresponding generating polynomials for $Z[\Lambda]^{\mathcal{W}}$. The generators of the fundamental Murray cone can be found by computing the highest weights of the polynomials.

3.2 MODULI SPACES FOR SMOOTH BPS MONOPOLES

For both singular and smooth monopoles we have identified the set of magnetic charges. This set always contains a subset closed under addition that arises by modding out Weyl transformations. On top of this we have seen that these sets are generated by a finite set of magnetic charges. This suggest that these generating charges correspond to a distinguished collection of basic monopoles and that all remaining magnetic charges give rise to multi-monopole solutions. By studying the dimensions of moduli spaces of solutions we can try to confirm this picture. In this section we shall only be concerned with smooth BPS monopoles. For such monopoles the magnetic charges satisfy the Murray condition.

3.2.1 FRAMED MODULI SPACES

The moduli spaces we shall discuss in this section are so-called framed moduli spaces. Such spaces are commonly used in the mathematically oriented literature on monopoles, see, for example, the book [51]. We shall discuss these spaces presently. In the next sections we review the counting of dimensions.

The moduli spaces we are considering correspond to a set of BPS solutions modded out by gauge transformations. The set of BPS solutions is restricted by the boundary condition we use, as discussed in section 2.3.3. Beside the finite energy condition one can use additional framing conditions, hence the terminology framed moduli spaces.

Recall from our discussion following (2.47) that the value $\phi(\hat{r}_0)$ of the asymptotic Higgs field at an arbitrarily chosen point $\hat{r}_0$ on the two-sphere at infinity determines the residual gauge group. It is therefore natural to restrict the configuration space to BPS solutions with $\phi(\hat{r}_0) = \Phi_0$ for a fixed value of Φ_0. The resulting space has multiple connected components labelled by the topological charge of the BPS solutions. This topological charge is given by the topological components m_i of $G_0 = G(\hat{r}_0)$ as explained in section 2.3. We shall thus consider the finite energy configurations satisfying the framing condition

$$\Phi(t\hat{r}_0) = \Phi_0 - \frac{G_0}{4\pi t} + \mathcal{O}\left(t^{-(1+\delta)}\right) \qquad t \gg 1, \tag{3.29}$$

where $\hat{r}_0$, Φ_0 and the topological components m_i of G_0 are completely fixed. The framed moduli space $\mathcal{M}(\hat{r}_0, \Phi_0, m_i)$ is now obtained from the configuration space by modding out certain gauge transformations that respect the framing condition. The full group of gauge transformations $\mathcal{G} : \mathbb{R}^3 \to G$ that respect this condition satisfy $\mathcal{G}(t\hat{r}_0) = h$ as $t \to \infty$ where $h \in H$. However, for the moduli space to be a smooth manifold one can only mod out a group of gauge transformations that acts freely on the configuration space. For example the configuration $\Phi = \Phi_0$ and $B = 0$ is left invariant by all constant gauge transformations given by $h \in H$. The framed moduli space is thus appropriately defined as the space of BPS solutions satisfying the boundary conditions (2.47) and (3.29), modded out by the gauge transformations that become trivial at the chosen base point $\hat{r}_0$ on the sphere at infinity.

The moduli space $\mathcal{M}(\hat{r}_0, \Phi_0, m_i)$ has several interesting subspaces which will play an important role in what is to come. These subspaces are related to the fact that there is a map f from the moduli space to the Lie algebra of G. This map is defined by assigning G_0 to each configuration. As explained in section 2.3.3 and 2.3.4, up to a residual gauge transformation G_0 is given by $G_0 = \frac{4\pi}{e} g \cdot H$ with g an element in the fundamental Murray cone. The topological components of g are of course fixed while the holomorphic charges are restricted by the topological charges. The image of f in the Lie algebra of G is thus a disjoint union of H orbits

$$C(g_1) \cup \cdots \cup C(g_n), \tag{3.30}$$

where g_i is the intersection of each orbit with the fundamental Murray cone. The map f defines a stratification of $\mathcal{M}(\hat{r}_0, \Phi_0, m_i)$. Each stratum $\mathcal{M}_{g_i}$ is mapped to a corresponding orbit $C(g_i)$ in the Lie algebra.
The remarkable thing about the stratification is that for a fixed topological charge the strata are disjoint but connected even though the images of the strata are disconnected sets in the Lie algebra of G. This follows from the fact that all BPS configurations in $\mathcal{M}(\hat{r}_0, \Phi_0, m_i)$ are topologically equivalent and can be smoothly deformed into each other. Under such smooth deformations the holomorphic charges can thus jump.
If the residual gauge group is abelian the stratification is trivial. Since the topological charges completely fix g there is only a single stratum $\mathcal{M}_g = \mathcal{M}(\hat{r}_0, \Phi_0, m_i)$.

There is another interesting moduli space we want to introduce. This so-called fully framed moduli space $\mathcal{M}(\hat{r}_0, \Phi_0, G_0) \subset \mathcal{M}(\hat{r}_0, \Phi_0, m_i)$ arises by imposing even stronger framing conditions. The points in the fully framed moduli space $\mathcal{M}(\hat{r}_0, \Phi_0, G_0)$ correspond to BPS configurations obeying the usual boundary conditions (2.47) and (3.29) but instead of only fixing Φ_0 we also choose a completely fixed magnetic charge G_0. Again the gauge transformations that become trivial at the chosen base point are modded out.
The fully framed moduli spaces have a special property in relation to the strata. Monopoles with magnetic charges G_0' and G_0 related by h in residual gauge group $H \subset G$ lie in the same stratum of the framed moduli space. Moreover, the action of $h \in H \subset G$ on the magnetic charges can be lifted to a gauge transformation $\mathcal{G} : S^2 \to G$ [44]. Since $\pi_2(G) = 0$ this gauge transformation can in turn be extended to a gauge transformation in $\mathbb{R}^3$ acting on the complete BPS solution. In other words the action of $h \in H$ on the Lie algebra can be lifted to an action on the framed moduli space such that each point in $\mathcal{M}(\hat{r}_0, \Phi_0, G_0)$ is mapped to a point in $\mathcal{M}(\hat{r}_0, \Phi_0, G_0)$. We thus see that all fully framed moduli spaces in a single stratum are isomorphic. In addition we also have that a stratum is nothing but a space of fully framed moduli spaces. Finally we conclude that locally we

must have that $M_i = C(g_i) \times M(\hat{r}_0, \Phi_0, G_0)$ where G_0 is defined by g_i.
If the residual gauge group is abelian the action of H on the magnetic charges is trivial. In this particular case the fully framed moduli space equals the single stratum and we have $\mathcal{M}(\hat{r}_0, \Phi_0, G_0) = \mathcal{M}(\hat{r}_0, \Phi_0, m_i)$.

3.2.2 Parameter counting for abelian monopoles

The dimensions of the framed moduli spaces for maximal symmetry breaking have been computed by Erick Weinberg [22]. From his index computation Weinberg concluded that there must be certain fundamental monopoles and that the remaining monopoles should be interpreted as multi-monopole solutions. The magnetic charges of these fundamental monopoles are precisely the generators of the Murray cone. Note that since there is no distinction between the Murray cone and the fundamental Murray cone in the abelian case we may also call these fundamental monopoles basic. Our conclusion is thus that the moduli space dimensions are consistent with the structure of the (fundamental) Murray cone. This result also holds in the non-abelian case albeit in a much less obvious way. To get some feeling for this general case we shall first briefly review Weinberg's results.

As before we consider a Yang-Mills theory with a gauge group G. The adjoint Higgs VEV μ is taken such that the gauge group is broken to its maximal torus $U(1)^r$, r is the rank of the group. In this abelian case the structure of the framed moduli as well as the structure of the Murray cone is relatively simple. Since there is no residual non-abelian symmetry there are no holomorphic charges. Consequently the magnetic charge is fully determined by the topological charges and the action of the residual gauge group on the magnetic charges is trivial. The fully framed moduli spaces thus coincide with the framed moduli spaces while the fundamental Weyl chamber of the Murray cone is identical to the complete cone. From the Murray-Singer analysis it follows that the stable magnetic charges are of the form:

$$g = \sum_{i=1}^{r} m_i \alpha_i^* \ , \quad m_i \in \mathbb{N}. \tag{3.31}$$

The r simple coroots α_i^* obviously generate the Murray cone and the positive expansion coefficients m_i can be identified with the topological charges as explained in section 2.3.3. According to the index calculations of Weinberg the dimensions of the moduli spaces are proportional to the topological charge:

$$\dim \mathcal{M}_g = \sum_{i=1}^{r} 4m_i. \tag{3.32}$$

As an illustration the Murray cone is depicted for $SU(3) \to U(1)^2$ in figure 3.2. For each charge the dimension of the moduli space is given. In general there are r indecomposable charges, one for each $U(1)$ factor. These basic monopoles all have unit topological charge. Thus we see that the dimension of the moduli space is proportional to the number $N = \sum m_i$ of indecomposable charges constituting the total charge. As Weinberg concluded this is precisely what one would expect for N non-interacting monopoles, and hence is seems consistent to view the higher topological charge solutions as multi-monopole solutions.

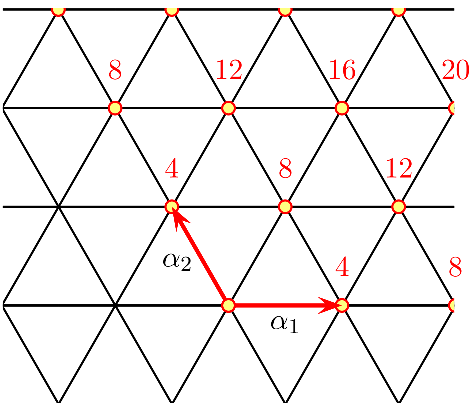


Figure 3.2: *The Murray cone for $SU(3)$ broken to $U(1) \times U(1)$. The generators of the cone are precisely the simple (co)roots α_1 and α_2 of SU(3). Both these charges correspond to unit topological charge in $\pi_1(U(1)^2) = \mathbb{Z} \times \mathbb{Z}$. All charges can be decomposed into the generating charges. The dimensions of the moduli spaces are proportional to the number of components. These dimensions are obviously additive.*

Before we continue with general symmetry breaking let us pause for moment to discuss the nature of the moduli space dimensions. These dimensions correspond to certain parameters of the BPS solutions. For the basic monopoles with charge α_i^* the obvious candidates for three of these are their spatial coordinates, i.e. the position of the monopole. The fourth is related to electric action by H_{α_i} which keeps the magnetic charge fixed but nevertheless acts non-trivially on the monopole solutions. This can be seen by considering exact solutions for the basic monopoles obtained by embedding $SU(2)$ monopoles [40, 22].

If the multi-monopole picture is correct the nature of the moduli space dimensions for higher topological charge is easy to guess. $3N$ correspond to the positions of the N constituents, while the remaining N dimensions arise from the action of the gauge group on the constituents. It has been shown by Taubes [52] that if $\sum m_i = N$ there exists an exact BPS solutions corresponding to N monopoles with unit topological charges. A similar result was obtained by Manton for two 't Hooft-Polyakov monopoles [53]. The positions of the individual monopoles can be chosen arbitrarily as long as the monopoles are well separated. This immediately confirms the given interpretation of the $3N$ parameters. Further evidence for this interpretation of the moduli space parameters can be found by studying the geodesic motion on the moduli space. For N widely separated monopoles the geodesic motion on the asymptotic moduli space corresponds to the motion of N dyons, considered as point-particles in $\mathbb{R}^3$, interacting via Coulomb-like forces. The conserved electric $U(1)$ charges appear in the geodesic approximation on the asymptotic moduli space because the metric has $U(1)$ symmetries. The correspondence between the classical theory on the asymptotic moduli space and the effective theory of classical dyons in space has up till now only been demonstrated for an arbitrary topological charge in a $SU(n)$ theory broken to $U(1)^n$ [54, 55, 56, 57, 58] and for topological charge 2 in an arbitrary theory with maximal symmetry breaking [59].

3.2.3 Parameter counting for non-abelian monopoles

Just as in the abelian case the dimensions of the framed moduli spaces for non-abelian monopoles are proportional to the topological charges. Hence the dimensions of the moduli spaces respect the addition of charges in the Murray cone. In that sense one could once more interpret monopoles with higher topological charges as multi-monopole solutions built out of monopoles with unit topological charges. This analysis would however ignore the fact that both the framed moduli space and the Murray cone have extra structure. The framed moduli space has a stratification while the magnetic charges have topological and holomorphic components. The holomorphic charges and thereby the strata are physically very important because they are directly related to the electric symmetry that can be realized in the monopole background as we shall discuss later in the section. Therefore one should wonder if these structures are compatible and if so how they will affect the multi-monopole interpretation.

The dimensions for the framed moduli spaces of monopoles have been computed by Murray and Singer for any possible residual gauge symmetry, either abelian or non-abelian [44]. Their computation does not rely on index methods but instead it is based on the fact that framed moduli spaces can be identified with certain sets of rational maps. Such a bijection was first proved by Donaldson for $G = SU(2)$ [60] and later generalized by Hurtubise and Murray for maximal symmetry breaking [61, 62, 63]. Finally the correspondence between framed moduli spaces and rational maps was proved for general gauge groups and general symmetry breaking by Jarvis [43, 64]. Murray and Singer have computed the dimensions of these spaces of rational maps. For further details we refer to the original paper. The $SU(n)$ case can also be found in [46].

One of the results of the calculations in [44] is that the dimension of the framed moduli space $\mathcal{M}(\hat{r}_0, \Phi_0, m_i)$ is given by:

$$\dim \mathcal{M}(\hat{r}_0, \Phi_0, m_i) = 4\sum_{i=1}^{s} (1 - 2\rho \cdot \alpha_i^*)\, m_i, \tag{3.33}$$

where ρ is the Weyl vector of the residual group and thus equals half the sum of the unbroken roots:

$$\rho = \frac{1}{2}\sum_{j=s+1}^{r} \alpha_j. \tag{3.34}$$

In equation (3.33) one sums over the broken roots and thus also over the topological charges.The dimensions of the framed moduli spaces have two important properties. First for $g = g' + g''$ with topological charges $m_i = m_i' + m_i''$ we have

$$\dim \mathcal{M}(\hat{r}_0, \Phi_0, m_i) = \dim \mathcal{M}(\hat{r}_0, \Phi_0, m_i') + \dim \mathcal{M}(\hat{r}_0, \Phi_0, m_i''). \tag{3.35}$$

Second if the residual gauge group equals the maximal torus $U(1)^r$ in G so that there are no holomorphic charges the dimension formula above reduces to Weinberg's formula

$$\dim \mathcal{M}(\hat{r}_0, \Phi_0, m_i) = 4\sum_{i=1}^{r} m_i. \tag{3.36}$$

We thus see that equation (3.33) for the dimension of the framed moduli space is a generalization of Weinberg's result. More importantly we find that dimensions of the framed moduli spaces respect the addition of charges in the Murray cone.

The dimensions of the framed moduli spaces are compatible with the addition of charges in the Murray cone. These dimensions do not depend on the holomorphic components. Naively it thus seems we can safely ignore these components. Nevertheless, from a physical perspective one is forced to take the holomorphic charge into account because it determines the allowed electric charge of a monopole as we shall discuss in a moment. It is thus very interesting to know how the holomorphic charges affects the fusion of single monopoles into multi-monopole configurations.

If we want to take the holomorphic charges into account we should consider the strata within the framed moduli spaces. These strata were introduced in section 3.2.1. For a given stratum G_0 is fixed up to the action of the residual gauge group and hence the holomorphic components of g are given up to Weyl transformations. The dimensionality of the stratum corresponding to g can be expressed in terms of the reduced magnetic charge as was shown by Murray and Singer [44].
Let g be any charge in the Murray cone and $\tilde{g}$ its reduced magnetic charge. Remember that $\tilde{g}$ is simply the lowest charge in the orbit of g under the action of the residual Weyl

group. The reduced magnetic charge can thus be expressed as:

$$\tilde{g} = \sum_{i=1}^{s} m_i \alpha_i^* + \sum_{j=s+1}^{r} h_j \alpha_j^*. \tag{3.37}$$

The dimensionality of the corresponding stratum $\mathcal{M}_g$ in the framed moduli space $\mathcal{M}(\hat{r}_0, \Phi_0, m_i)$ is given by:

$$\dim \mathcal{M}_g = \sum_{i=1}^{s} 4m_i + \sum_{j=s+1}^{r} 4h_j + \dim C(\Phi_0) - \dim C(\Phi_0) \cap C(G_0). \tag{3.38}$$

$C(\Phi_0) \in G$ is the centralizer subgroup of the Higgs VEV, i.e. it is simply the residual gauge group H. Similarly, $C(G_0) \in G$ is the centralizer of the magnetic charge. Hence the fourth term in the equation above equals the dimensionality of the subgroup in H that leaves G_0 invariant. So the last two terms in equation (3.38) express the dimension of the orbit of the magnetic charge G_0 under the action of the residual gauge group.
In figure 3.3 we have worked out formula (3.38) for $SU(3) \to U(2)$ for each charge in the Murray cone. In this particular case the H orbits of the magnetic charges are either 2-spheres or they are trivial.

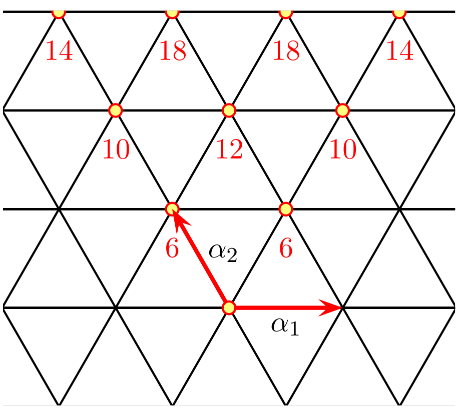


Figure 3.3: *Dimensions for the strata of the framed moduli spaces for $SU(3)$ broken to $U(2)$.*

The next goal is to relate the dimensions of the strata to the generators of the Murray cone found in section 3.1.1, the monopoles with unit topological charges. In the abelian case discussed previously such a relation is obvious. Since there are no stratifications the moduli space dimensions are proportional to the topological charges. In the true non-abelian case such a simple relation is distorted by the centralizer terms in formula (3.38). This is easy to see in the $SU(3)$ example in figure 3.3. Therefore we shall have to leave these centralizer terms out in our analysis. Since the centralizer terms correspond to the orbit of the magnetic charges under the action of residual gauge group, discarding the centralizer terms amounts to restricting to the fully framed moduli spaces introduced in section 3.2.1.

There are good arguments to discard the centralizer terms in the present discussion or at least to treat them on a different footing than the remaining terms in (3.38). The centralizer terms count the dimensions of the orbit of the magnetic charge under the action of the electric group. Naively one would thus expect that this orbit is related to the electrical properties of the monopoles. Such a picture is flawed because already at the classical level there is a topological obstruction for implementing the full residual electric group $H \subset G$ globally as has been proven by various authors [24, 26, 25, 27, 28]. This obstruction is directly related to the fact that a magnetic monopole defines a non-trivial H bundle on a sphere at infinity. A subgroup $H' \subset H \subset G$ is implementable as a global symmetry in the background of a monopole if the transition function (2.57)

$$\mathcal{G}(\varphi) = \exp\left(\frac{ie}{2\pi} G_0 \varphi\right) \tag{3.39}$$

is homotopic to a loop in

$$Z_H(H') = \{h \in H : hh' = h'h \ \ \forall h' \in H\} \tag{3.40}$$

the centralizer of $H' \subset H$. Note that the maximal torus $U(1)^r \subset H$ is always implementable. As as rule of thumb one finds that H' can be non-abelian if up to unbroken coroots the magnetic charge has one or more vanishing weights with respect to the non-abelian component of H. This follows from the fact that the holomorphic components of the magnetic charge are not conserved under smooth deformations.
There is an even stronger condition on the electric symmetry that can be realized in the monopole background. One can show [26] that the action of the residual electric group maps finite energy configurations to monopole configurations with infinite energy if the magnetic charge is not invariant. The interpretation is that all BPS configurations with finite energy whose magnetic charges lie on the same electric orbit are separated by an infinite energy barrier.
Classically one thus finds that only if the generators of the residual gauge group H commute with the magnetic charge one can define a global rigid action of H. In other words the monopole effectively breaks the symmetry further down so that only the centralizer group can be realized as a symmetry group. For example in the case that $SU(3)$ is broken to $U(2)$ monopoles with magnetic charge $g = 2\alpha_2$ can only carry electric charges under $U(1)^2$, while monopoles in the same framed moduli space with $g = 2\alpha_2 + \alpha_1$ might carry charges under the full residual $U(2)$ group. These obstructions persist at the semi-classical level [23, 24, 65].

The dimensions of the fully framed moduli spaces have a simple expression in terms of the topological and holomorphic components of the reduced magnetic charge $\tilde{g}$ [44]:

$$\dim \mathcal{M}(\mu, \Phi_0, G_0) = \sum_{i=1}^{s} 4m_i + \sum_{j=s+1}^{r} 4h_j. \tag{3.41}$$

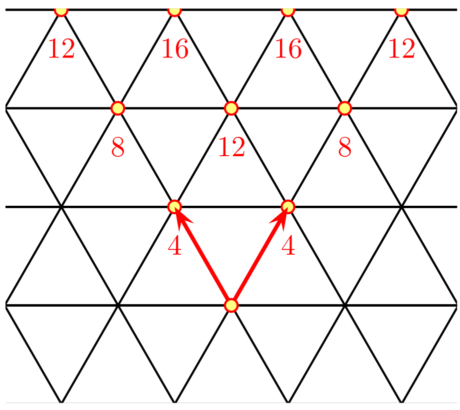


Figure 3.4: *Dimensions for the fully framed moduli spaces for $SU(3)$ broken to $U(2)$, and the generators of the Murray cone. The dimensions are only additive if one moves along the central axis of the cone or away from it.*

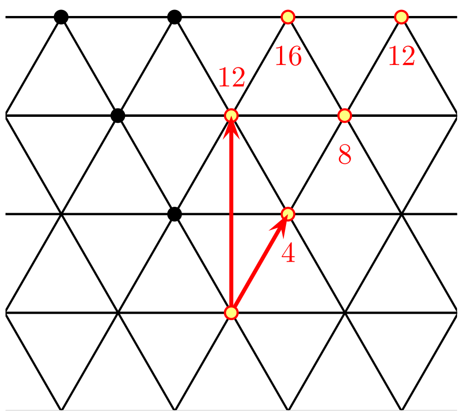


Figure 3.5: *The fundamental Murray cone for $SU(3)$ broken to $U(2)$. In this example the magnetic charge lattice is interpreted as the weight lattice of $U(2)$. The fundamental Murray cone is the intersection of the full cone with the fundamental Weyl chamber of the $U(2)$ weight lattice. The dimensions of the fully framed moduli spaces are additive under the composition of the generators depicted by the arrows.*

Previously we have found that the Murray cone is spanned by the magnetic charges with

unit topological charges. We might hope that the dimensions of the fully framed moduli spaces behave additively with respect to the expansion into these indecomposable charges as was the case for the framed moduli space. Such additive behaviour does indeed occur, but only partially. For instance the case of $SU(3) \to U(2)$ is worked out in figure 3.4. The additivity of the moduli space dimensions still holds as long as we stick to one of the Weyl chambers of the cone, defined with respect to the residual Weyl action.
Apparently the dimensions of the fully framed moduli spaces are not compatible with the Murray cone in general. However, as we will prove below these dimensions are compatible with the fundamental Murray cone.
In the abelian case this is obviously true. The Weyl group of the residual group is now trivial and there is no additional identification within the cone. Therefore we can refer back to the previous sections where we found that the generating charges have unit topological charge and that the dimensionality of the moduli space is proportional to the total topological charge. Our favourite example in the truly non-abelian case $SU(3) \to U(2)$ is worked out in figure 3.5. The generators of the fundamental Murray cone are easily recognized and the additivity of dimensions is easily confirmed.
We claim that the additivity of the moduli space dimensions with respect to a decomposition in generating charges of the fundamental Murray cone holds in general. Without an explicit set of generators it seems we cannot prove this directly. However, it suffices to check the additivity for every pair of charges in the fundamental cone.

Proposition 3.4 *For any pair of magnetic charges g and g' in the fundamental Murray cone we have for the fully framed moduli spaces* $dim\,\mathcal{M}_g + dim\,\mathcal{M}_{g'} = dim\,\mathcal{M}_{g+g'}$.

Proof. Recall from equation (3.41) that the dimensions of the fully framed moduli space are proportional to the topological and holomorphic charges of the reduced magnetic charge. We thus have to show that the topological and holomorphic charges add. These charges are given by the inner product of the reduced magnetic charge with respectively the broken and unbroken fundamental weights as explained in section 2.3.3 and 2.3.4. For example $m_i = \lambda_i \cdot \tilde{g}$. Next we note that there exists a Weyl transformation $w \in W(H) \subset W(G)$ that maps the fundamental Weyl chamber to the anti-fundamental Weyl chamber. Thus if g, g', g'' lie in the fundamental Murray cone and $g'' = g + g'$ the reduced magnetic charges satisfy $\tilde{g}'' = w(g'') = w(g) + w(g') = \tilde{g} + \tilde{g}'$. As a last step we find that $m_i'' = \lambda_i \cdot \tilde{g}'' = \lambda_i \cdot (\tilde{g} + \tilde{g}') = m_i + m_i'$. Similar results hold for the holomorphic charges. □

The dimensions of the fully framed moduli spaces only respect the addition of charges in the Murray cone if the charges are restricted to one Weyl chamber, for example the fundamental Weyl chamber. This is consistent with our conclusion at the end of section 2.3.4 that the magnetic charge sectors are labelled by weights in the fundamental Weyl chamber of the residual dual group.

In this section we have established a non-abelian generalization of Weinberg's analysis for abelian monopoles: we have shown that the dimensions of the fully framed moduli spaces respect the addition of magnetic charges within the fundamental Murray cone. Just as Weinberg we are now led to the conclusion that there is a distinguished set of basic monopoles. The charges of these basic monopoles correspond to the generators of the fundamental Murray cone. The remaining charges in the fundamental Murray cone are then associated with multi-monopole solutions.

For maximal symmetry breaking the set of basic monopoles coincides with the monopoles with unit topological charge. In our proposal this is not true in the general case. There can be basic monopoles with non-minimal topological charges. In the next section we shall discuss additional evidence to support our conclusion that basic monopoles are always indecomposable, even if they have non-minimal topological charges.

3.3 Fusion properties of non-abelian monopoles

In the previous sections we argued that smooth BPS monopoles with non-trivial charges can consistently be viewed as multi-monopole solutions built out of BPS configurations with minimal charges. These classical fusion rules cannot always be verified directly because of the complexity of the BPS equations. In this chapter we have therefore gathered all available circumstantial evidence. These consistency checks can be organized into four different themes: the existence of generating charges and the consistent counting of moduli space parameters have been discussed in the previous sections. Below in section 3.3.1 and 3.3.2 we shall study some examples where one can verify the classical fusion rules directly. For singular BPS monopoles there is a similar set of generating charges, a consistent counting of parameters and a consistent way to patch classical solutions together as we discuss in section 3.3.3. These analogies form a remarkable hint suggesting that the classical fusion rules we have found for smooth BPS monopoles are indeed correct. Finally in section 3.3.4 we look ahead and discuss how this analogy between singular and smooth BPS monopoles might help us to derive the semi-classical fusion rules of smooth BPS monopoles and conversely how to get a better understanding of the generalized electric-magnetic fusion rules in the singular case.

3.3.1 Patching smooth BPS solutions

The first hint revealing the existence of multi-monopole solutions built out of certain minimal monopoles comes from the fact that there is a small set of indecomposable charges generating the full set of magnetic charges. In this section we use results of Taubes obtained in [52] to show that certain monopoles with non-trivial charges are indeed multi-monopole solutions respecting the decomposition of the magnetic charge into generating charges. We shall first discuss maximal symmetry breaking. In this case all monopoles with higher topological charges are manifestly seen to be multi-monopoles. Second we shall deal with non-abelian residual gauge groups. In this case Taubes' result gives a consistency check for the classical fusion rules.

For maximal symmetry breaking the set of magnetic charges corresponds to the Murray cone and is generated by the broken simple coroots. For each of these coroots an exact solution is known. These are spherically symmetric $SU(2)$ monopoles [40, 22, 65]. For G equal to $SU(2)$ one has the usual 't Hooft-Polyakov monopole [6, 7], while for higher rank gauge groups one can embed 't Hooft-Polyakov monopoles via the broken simple roots. Since these monopoles have unit topological charges they are manifestly indecomposable.

Exact solutions are also known in other cases. It was shown by Taubes [52] that there are solutions to the BPS equation for any charge $g = m_i \alpha_i^*$ with $m_i > 0$. Hence for all

charges in the Murray cone solutions exist. These solutions are constructed out of superpositions of embedded $SU(2)$ monopoles. The constituents are chosen such that the sum of the individual charges matches the total charge. These solutions become smooth solutions of the BPS equations if the constituents are sufficiently separated. This proves that for all magnetic charges with higher topological charges multi-monopole solutions exist.
One might wonder if all solutions with higher topological charges are indeed multi-monopole solutions. For any given topological charge the framed moduli space is connected. Thus any point in this moduli space is connected to another point corresponding to a widely separated superposition as described by Taubes. Any monopole configuration can thus be smoothly deformed so that the individual components are manifest. This does indeed show that any smooth abelian BPS monopole can consistently be viewed as multi-monopole configuration built out of indecomposable monopoles.

The multi-monopole picture above for maximal symmetry breaking can be generalized to arbitrary symmetry breaking. For any given topological charge there exist smooth solutions of widely separated monopoles. According to Taubes the building blocks of these smooth solutions correspond to the $SU(2)$ monopoles embedded via the broken simple roots. The framed moduli space does not depend on the full magnetic charge, but only on the topological components. Moreover the framed moduli space is always connected. We now find that any solution of the BPS equation with higher topological charges can be deformed to a configuration which is manifestly a multi-monopole solution.
However, this decomposition via widely separated multi-monopole solutions does not respect the additive structure of the Murray cone unless we completely ignore the holomorphic charges. In the previous section we argued that this does not make sense from a physical perspective because the allowed electric excitations depend on the holomorphic charges. Moreover we have found that if we take these holomorphic charges into account we should restrict the magnetic charges to lie in the fundamental Murray cone. The appropriate moduli spaces to consider in this situation are the fully framed moduli spaces. The question now is if these fully framed moduli spaces contain configurations which can be interpreted as widely separated monopoles.
With the results of Taubes we can answer this question unambiguously for one of the fully framed moduli spaces in the set of spaces defined by the topological charge. We start out with a magnetic charge that is equal to a sum of unbroken simple coroots. Such a charge does not lie in the fundamental Murray cone, but instead in the anti-fundamental Weyl chamber of the Murray cone. This implies that there is a Weyl transformation that maps such a magnetic charge to the fundamental Murray cone. Similarly, there is a related large gauge transformation that maps Taubes' multi-monopole configurations to new solutions of the BPS equation. These transformed configurations are again widely separated superpositions, but now the buildings blocks correspond to $SU(2)$ monopoles embedded via the Weyl transformed broken roots. Note that magnetic charges of these constituents are

precisely the generators of the fundamental Murray cone with unit topological charges. We thus obtain the following result: let g equal a sum of generators of the fundamental Murray cone with unit topological charges. The fully framed moduli space corresponding to g has a subset of configurations that are manifestly multi-monopole solutions. Since the fully framed moduli space is connected any monopole with charge g can be interpreted as a multi-monopole solution.

The considerations above only involved indecomposable charges corresponding to simple coroots. For a maximally broken gauge group this is sufficient to provide convincing evidence for the multi-monopole picture. In the non-abelian case one should also take other generators into account. We will come back to this in the next section.

3.3.2 MURRAY CONE VS FUNDAMENTAL MURRAY CONE

One problem we encounter in this thesis is that it is not completely clear what the full set of magnetic charges is supposed to be, either the Murray cone or the fundamental Murray cone. By the same token it is a priori not clear what the truly indecomposable monopoles are, the fundamental monopoles or the basic monopoles. The fundamental Murray cone is slightly favoured because the large gauge transformations have been modded out. On the other hand not all generating charges of the fundamental Murray cone have unit topological charges, while the fundamental monopoles generating the Murray cone do. This suggest that the monopoles corresponding to the generators of the fundamental Murray cone, the basic monopoles might be decomposable into fundamental monopoles related to the generators of the Murray cone. What seems to settle this issue though is that there is only a consistent counting of moduli space parameters if we restrict to the fundamental Murray cone. The indecomposability for the basic monopoles with non-minimal topological charges can be understood from the existence of so-called cloud parameters. These clouds emerge as soon as one attempts to split a basic monopole into fundamental monopoles. Below we explain this for the case that $G = SU(3)$ is broken to $U(2)$.

The case $SU(3) \rightarrow U(2)$ is an interesting example to discuss issues regarding composition and decomposition because some of the corresponding non-trivial moduli spaces have been thoroughly investigated. Specifically for $g = 2\alpha_2 + \alpha_1$, one of the generators of the fundamental Murray cone, the 12 dimensional fully framed moduli space and its metric have been found Dancer [66]. Determining the isometries of the metric reveals the nature of almost all of the 12 parameters. Three parameters are related to $\mathbb{R}^3$, the center of mass position in space. The action of $U(2) \times SO(3)$ shows the presence of three rotational degrees of freedom and four large gauge modes. After the removal of translations, gauge freedom and rotations one is left with a two-dimensional space. This space turns out to be parameterized by k and D with $0 \leq k \leq 1$ and $0 \leq D < \frac{2}{3}K(k)$, where

$K(k)$ denotes the first complete elliptic integral $K(k) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2(\theta))^{-1/2} d\theta$ [66]. The interpretation of these two parameters seems somewhat mysterious. To understand their significance, the behaviour of the BPS solutions have been studied numerically for various values of these parameters [67, 68, 69]. See section 8 of [70] and section III.B of [71] for a review.
There is a subset of solutions where the energy density has two maxima symmetrically positioned about the center of mass. If the parameter D is increased the peaks of the energy density becomes more pronounced and move further from the center of mass. This seems to indicate that certain solutions can be viewed as a pair of widely separated particles. The question that comes to mind now is the following: do these widely spaced lumps correspond to a pair of monopoles with unit topological charge? In this particular case there is only one broken simple root, and hence there is only one class of embedded $SU(2)$ monopoles with unit topological charge giving rise to a 4 dimensional fully framed moduli space. Taubes has shown that such solutions can be patched together yielding widely separated solutions with higher topological charges [52]. A pair of these patched solutions with magnetic charge $\tilde{g} = \alpha_2 + \alpha_1$ would give a configuration with total charge $2\alpha_2 + 2\alpha_1$. We thus see that these solutions do not lie in the 12 dimensional Dancer moduli space but in the neighbouring 8 dimensional fully framed moduli space. Before drawing any conclusions we recall one subtle point. All monopoles of equal topological charge lie in one connected moduli space, which is divided up into strata. By dividing out large gauge transformations these strata reduce to the fully framed moduli spaces we discussed here. To be more precise the 8 dimensional fully framed moduli space related to $g = 2\alpha_2 + 2\alpha_1$ lies in a 10 dimensional stratum which is the boundary of the Dancer moduli space. A measure for the distance to the boundary of the Dancer moduli space is given by $1/a$ where $a = D/(K(k) - \frac{3}{2}D)$. The widely separated monopoles in the Dancer moduli space, and as a matter of fact any configuration in the Dancer moduli space can be deformed into widely separated monopoles discussed by Taubes. As we explained in section 3.3.1 this decomposition does not respect the additive structure of the Murray cone nor the addition of charges in the fundamental Murray cone. What is even more striking is that this deformation will give rise to highly non-localized degrees of freedom in the form of a non-abelian cloud.

A rather insightful computation to illustrate the appearance of the non-abelian cloud as one moves to the boundary of the Dancer moduli space is discussed by Irwin [69]. In his paper Irwin computes the asymptotic behaviour for the magnetic field of axially symmetric trigonometric monopoles ($k = 0$) in the Dancer moduli space as $a \to \infty$. In the string gauge the asymptotic fields in this limit are given by:

$$\begin{aligned}
\Phi &= \Phi_0 - \frac{1}{er}t_0 - \frac{1}{er(1+r/a)}t_3 \\
*F &= \frac{1}{er^2}t_0 dr + \frac{1+2r/a}{er^2(1+r/a)^2}t_3 dr + \frac{1}{ear^2(1+r/a)^2}\left(t_1 d\theta - \sin\theta t_2 d\varphi\right) \qquad (3.42) \\
A &= -\frac{1}{e}\cos\theta\,(t_0+t_3)\,d\varphi - \frac{1}{e(1+a/r)}\left(t_2 d\theta + \sin\theta t_1 d\varphi\right).
\end{aligned}$$

The expectation value Φ_0 is proportional to t_0. The matrices t_i are the generators of the residual gauge group $U(2) \subset SU(3)$: $t_0 = (\alpha_1 + 2\alpha_2)\cdot H = \sqrt{3}H_2$, $t_1 = \frac{1}{2}\left(E_{\alpha_1} + E_{-\alpha_1}\right)$, $t_2 = -\frac{i}{2}\left(E_{\alpha_1} - E_{-\alpha_1}\right)$ and $t_3 = \alpha_1 \cdot H = H_1$. With these conventions the commutation relations for the $SU(2)$ generators are given by $[t_i, t_j] = i\epsilon_{ijk}t_k$. Obviously we also have $[t_0, t_i] = 0$. As a side remark we note that (3.42) gives an exact solution of the BPS equations for $U(2)$ which is singular at r=0.
The behaviour of the solution above shows that the non-abelian fields penetrate outside the core of the monopoles up to some finite distance determined by the parameter a. Beyond this distance the magnetic field becomes abelian, which is in agreement with the boundary condition at infinity. The interpretation of this observation is that the monopoles are surrounded by a non-abelian cloud screening the non-abelian charge. As one moves all the way to the boundary of the Dancer moduli space and a becomes infinite the cloud gets diluted so that the non-abelian field yields to infinity resulting in non-vanishing holomorphic charges.

This whole exposition does lead us to an important conclusion: the behaviour of the Dancer monopoles shows us that these configurations can indeed be split up into separate lumps. At the same time this separation yields a non-localized degree of freedom. Since this cloud parameter does not correspond to one monopole or the other, the two widely spaced lumps are not the same as they would be on their own. Therefore the Dancer monopoles cannot be decomposed. In this particular example we thus see that basic monopoles are indeed indecomposable. We expect that similar arguments should hold in general.

3.3.3 PATCHING SINGULAR BPS SOLUTIONS

There are striking similarities between the results obtained in this thesis for smooth BPS monopoles and results obtained by Kapustin and Witten regarding singular BPS monopoles [18]. In the context of singular monopoles we shall discuss the existence of fundamental and basic monopoles, consistent counting of moduli space parameters, patching of classical solutions and the indecomposability of basic monopoles with non-trivial topological charges.

The magnetic charge lattice for singular monopoles in a gauge theory with gauge group H is determined by the Dirac quantization condition. As we reviewed in section 2.3.1 this lattice can be identified with the weight lattice $\Lambda(H^*)$ of the GNO or Langlands dual group. The magnetic weight lattice contains an important subset, the set of magnetic charge sectors, which is obtained by modding out the Weyl group of H^*. This subset can thus be identified with the fundamental Weyl chamber in $\Lambda(H^*)$. Modding out the Weyl group of H^* is natural because a magnetic charge $\lambda \in \Lambda(H^*)$ is only defined up to Weyl transformations. Note that these Weyl transformations acts as large gauge transformations on the BPS solutions.

The existence of generating charges within the weight lattice $\Lambda(H^*)$ of the dual gauge group and within its fundamental Weyl chamber has been discussed in sections 3.1.2 and 3.1.3. These generating charges correspond to what we define as respectively fundamental monopoles and basic monopoles. The basic monopoles, not the fundamental monopoles, form the building blocks of singular multi-monopole solutions of the BPS equations just as we concluded for smooth BPS solutions. This is seen indirectly by analyzing the moduli space parameters.

The moduli spaces for singular BPS monopoles introduced by Kapustin and Witten are spaces of so-called Hecke modifications and correspond to orbits in the affine Grassmannian. For further details we refer to [18] and references therein. It is important that these moduli spaces are labelled by a dominant integral weight in the weight lattice of the dual gauge group H^*. We also note that these moduli spaces are closed under large gauge transformations, hence magnetic charges on one Weyl orbit correspond to the same moduli space. For completeness we mention that the compactifications of these moduli spaces are singular. The singular subspaces in $\overline{\mathcal{M}}_\lambda$ correspond to the moduli spaces $\mathcal{M}_{\lambda'}$ where $\lambda' < \lambda$.

The dimensionality of an orbit $\mathcal{M}_\lambda$ in the affine Grassmannian labelled by a dominant integral weight $\lambda \in \Lambda(H^*)$ is given by [72]:

$$\dim \mathcal{M}_\lambda = 2\lambda \cdot \rho, \tag{3.43}$$

where ρ is the Weyl vector of H and thus equals half the sum of the simple roots of H, see e.g. [73] for a brief summary. We now immediately find for a pair of dominant integral weights λ and λ':

$$\dim \mathcal{M}_\lambda + \dim \mathcal{M}_{\lambda'} = \dim \mathcal{M}_{\lambda+\lambda'}. \tag{3.44}$$

Here we use the fact that sum of two dominant integral weights is again a dominant integral weight. We thus see that the moduli space dimensions respect the addition of charges in the fundamental Weyl chamber of H^*. It is not difficult to see that these dimensions are not consistent with the addition of charges in the complete weight lattice of H^*. Similar results where obtained in section 3.2.3 for the Murray cone and the fundamental Murray cone.

The formalism in which Kapustin and Witten work is so powerful that one can quite explicitly see that all singular monopoles with non-basic charges are indeed multi-monopole solutions. This is related to the fact that the singularities of the compactified moduli spaces can be blown-up in a very specific way. We briefly sketch how this works. Singular BPS monopoles correspond to 't Hooft operators which create the flux of a Dirac monopole with a singularity at a point $p \in \mathbb{R}^3$. These 't Hooft operators can in turn be identified with Hecke operators. The Hecke operators act on vector bundles over $\mathbb{C}$ in this case in such a way that the trivialization outside a preferred point on $\mathbb{C}$ is respected. To achieve this relation $\mathbb{R}^3$ is identified with $\mathbb{C} \times \mathbb{R}$. For a non-zero magnetic charge the Hecke operator maps a trivial bundle to a non-trivial bundle. These two bundles over $\mathbb{C}$ are identified with pullback bundles of the non-trivial bundle over $\mathbb{R}^3 \setminus \{p\}$ corresponding to the singular BPS configuration. The two embeddings of $\mathbb{C}$ are chosen at opposite sides of $p \in \mathbb{C} \times \mathbb{R}$. Note that the isomorphism class of the resulting modified bundle does not only depend on the magnetic charge but also in a certain way on the trivialization of the trivial bundle one started out with. This why one Hecke operator gives rise to a space of Hecke modifications.
A relevant but actually not very deep observation is that all Hecke operators can be decomposed as a sequence of basic Hecke operators and thus all 't Hooft operators can be decomposed as sequence of basic 't Hooft operators. These basic operators create the flux of a Dirac monopole associated to a basic charge in the fundamental Weyl chamber of H^*. An important as well as deep consequence of the identification of 't Hooft operators and Hecke operators is that the resulting sequence of basic 't Hooft operators can be separated in space. Each basic 't Hooft operator is positioned between two copies of $\mathbb{C} \subset \mathbb{R}^3$ and the associated Hecke operators map one bundle over $\mathbb{C}$ to the other bundle at the reverse side of the singularity. The resulting bundles over $\mathbb{C}$ can be considered as a series of pullback bundles in a bundle over $\mathbb{R}^3$ corresponding to a series of smoothly patched BPS solutions.

Singular BPS solutions corresponding to basic monopoles may have non-trivial topological charges just as we have seen for smooth BPS solutions. One might again wonder if such basic monopoles can be split up into fundamental monopoles which do have unit topological charges. Intuitively this does not seem difficult. One would expect that there exists an exact multi-monopole solution of widely separated fundamental monopoles in the same topological sector. Because all spaces of Hecke modifications in one topological sector are connected one can now deform the original monopole into a manifest multi-monopole solution. Just as for smooth monopoles this deformation does not respect the holomorphic charges. There is also a more subtle way to look at this holomorphic ob-

struction.
As an example we consider $H^* = U(2)$. This case has been worked out in quite some detail by Kapustin and Witten. The basic monopole with unit topological charge corresponds to the highest weight λ of the fundamental representation of $U(2)$. The basic monopole with topological charge equal to 2 has a magnetic charge given by $2\lambda - \alpha$, where α is the simple root of $SU(2)$. The compactification of $\mathcal{M}_{2\lambda}$ is given by the singular space $\mathbb{WCP}_2(1,1,2)$. The singularity corresponds to the 0-dimensional space $\mathcal{M}_{2\lambda-\alpha}$. The singularity of $\mathbb{WCP}_2(1,1,2)$ can be blown-up to obtain $\mathbb{CP}_1 \times \mathbb{CP}_1$. Since $\mathbb{CP}_1 = \mathcal{M}_\lambda$ the blow-up obviously gives the moduli space of two separated fundamental monopoles. On thus sees that the a basic monopole with topological charge 2 can be deformed into two separate monopoles only if one attributes extra degrees of freedom by embedding the moduli space as a singularity in a larger space and only if these degrees of freedom are changed in a non-trivial way by blowing-up the singularity. It follows for $U(2)$ that classically basic monopoles are not truly separable into fundamental monopoles. For general monopoles similar arguments should hold.
Note that for smooth monopoles we used the emergence of non-localized degrees of freedom to show that the basic $U(2)$-monopoles are indeed indecomposable. Though the motivation via clouds is quite different from the argument used right above in the singular case the result is the same: the charges of the indecomposable monopoles in the smooth theory are subset of the indecomposable charges for singular monopoles. In that sense the classical fusion rules for smooth BPS monopoles are consistent with those for singular BPS monopoles.

3.3.4 Towards semi-classical fusion rules

In the literature it has often been assumed that the BPS solutions corresponding to weights of the fundamental representation of H^* give rise to a single H^*-multiplet [40, 74, 75, 76, 77, 70, 78] as would be favourable to the conjecture that these monopoles can be regarded as massive gauge particles of the dual theory. This proposal runs into trouble in particularly for non-simply laced gauge groups because the electric action on the classical solutions does clearly not commute with the magnetic action of the residual dual group on the magnetic charges. From the classical fusion rules we find that smooth BPS solutions, and actually also singular BPS solutions, are labelled by dominant integral weights in the weight lattice of the residual dual group H^*. This suggest that each electric orbit in the magnetic charge lattice of H^* thus gives rise to a unique H^*-multiplet. This form of the GNO duality conjecture has been proven by Kapustin and Witten in the case of singular BPS monopoles [18]. They show that the semi-classical fusion rules for singular BPS monopoles are indeed the fusion rules of H^*. Since the classical fusion rules for singular and smooth BPS monopoles are completely analogous and because the semi-classical fusion rules must also agree one can expect that a similar approach can be used to derive the semi-classical fusion rules in the smooth case. It is not immediately clear though how such a program can be realized and some major hurdles have to be overcome. We shall discuss this shortly.

In the Kapustin-Witten approach the semi-classical fusion rules are found from the quantum mechanics on the moduli spaces. A similar strategy but with a less ambitious goal in mind was adopted in the case of smooth monopoles by Dorey et al. in [77]. These authors tried to give a consistent counting of states, an attempt that turned out not to be completely successful. In hindsight we can understand that the problem was caused by the fact Dorey et al. did not use the same moduli spaces as Kapustin and Witten. The moduli spaces used by Kapustin and Witten can be identified with orbits in the affine Grassmannian labelled by the magnetic charges in the weight lattice of H^*. These orbits do contain the orbits of the magnetic charge in $\mathfrak{h}$, the Lie algebra of H, under the action of the gauge group H as used in [77]. Only if the magnetic charge labels a so-called minuscule representation the orbit in the affine Grassmannian is isomorphic to the orbit in the Lie algebra of H. In these cases the number of ground states of the quantum mechanics on the orbit in the Lie algebra agrees with the dimension of the irreducible representation labelled by the magnetic charge. In other cases the orbit in $\mathfrak{h}$ is a non-trivial subspace within the orbit in the affine Grassmannian. The degeneracy of the ground state of the quantum mechanics on the orbit in $\mathfrak{h}$ under-estimates the dimension of the magnetic representations.

If one wants to retrieve a counting of states consistent with the irreducible representations of H^* as well as the fusion rules of H^* one is forced to consider the orbits in the affine Grassmannian. The problem is that it is only partially clear how these magnetic moduli spaces are to appear within the full moduli spaces of smooth BPS monopoles. What is consistent though is that the orbits of the magnetic charges under the electric action, which are part of the related orbits in the affine Grassmannian, have to be treated on a different footing within the framed moduli spaces as we discussed in section 3.2.3.
Another hint that shows the relevance of the affine Grassmannian for smooth monopoles comes from considering non-abelian vortex solutions. In particular the moduli space of axially symmetric vortices with winding number 2 in a certain $U(2)$ gauge theory equals $\mathbb{WCP}_2(1,1,2)$ [79]. These vortex solutions can be identified with a co-axial composite of two fundamental vortices and the moduli space reduces to $\mathbb{CP}_1 \times \mathbb{CP}_1$ in the widely separated situation. Each of these $\mathbb{CP}_1$-factors corresponds to the moduli space of a fundamental vortex. As the vortex and monopole moduli are related by a homotopy matching argument [80] it seems natural to assume that the $\mathbb{WCP}_2(1,1,2)$ is the appropriate moduli space for smooth monopoles with topological charge 2 in a theory where the gauge group $SU(3)$ is broken down to $U(2)$.

One of the drawbacks of the approach used by Kapustin and Witten is that it is not clear how to deal with electric excitations of magnetic monopoles even though one can introduce general Wilson-'t Hooft operators in for example certain $\mathcal{N} = 2$ gauge theories [81]. If one manages for smooth monopoles to introduce the appropriate magnetic moduli spaces in combination with the fully framed moduli spaces then one obtains an interesting model to study electric-magnetic symmetry. This would be an important achievement because it is not known what this unified electric-magnetic symmetry is. It is clear though that this symmetry is not the group $H \times H^*$ as originally proposed in [2] because the magnetic charge effectively breaks the electric group as we discussed in section 3.2.3. In [78] a unified electric-magnetic group was introduced for $H = U(n)$ which does respect this interaction between electric representations and magnetic charges. We will eleborate on this in chapter 4.

CHAPTER 4

FUSION RULES FOR DYONS

In this chapter we try to find an answer to the question what the electric-magnetic symmetry in a non-abelian gauge theory amounts to. This question may be paraphrased in many ways, varying in physical content and mathematical sophistication. Our main goal is to find a consistent large distance description of the electric, magnetic and dyonic degrees of freedom. We would like to uncover the hidden algebraic structure which governs the labelling and the fusion rules of the charge sectors in general gauge theories.

Even though electric degrees fall into irreducible representations of the group G and magnetic sectors correspond to irredicible representations of the dual group G^*, dyonic charge sectors are not labelled by a $G \times G^*$ representation. Since in a give monopole background the electric symmetry is restricted to the centraliser in G of the magnetic charge as discussed in the previous chapter, dyonic sectors are instead characterised (up to gauge transformations) by a magnetic charge and an electric centraliser representation. As we review in section 4.2 there is an equivalent labelling of dyonic charge sectors by elements in the set $(\Lambda \times \Lambda^*)/\mathcal{W}$, where $\mathcal{W}$ is the Weyl group which isomorphic for G and G^* and Λ and Λ^* are the weight lattices of respectively G and G^*.
In section 4.3 we introduce a candidate for a unified electric-magnetic symmetry group in Yang-Mills theory, which we call the skeleton group. A substantial part of this section is taken up by a detailed exposition of various aspects of the skeleton group which are needed in subsequent sections. The first important result of this chapter is that in section 4.4 we provide evidence for the relevance of the skeleton group by relating the representation theory of the skeleton group to the labelling and fusion rules of charge sectors. In particular we show that the labels of electric, magnetic and dyonic sectors in a non-abelian Yang-Mills theory can be interpreted in terms of irreducible representations of the skeleton group. Decomposing tensor products of irreducible representations of the skeleton

group thus gives candidate fusion rules for these charge sectors. We demonstrate consistency of these fusion rules with the known fusion rules of the purely electric or magnetic sectors, and extract new predictions for the fusion rules of dyonic sectors in particular cases.
One should expect the dyonic sectors and fusion rules to be robust and in particular independent on the dynamical details of the particular model. Hence, in this chapter we will not focus on special models. Nonetheless our results must be consistent with what is known for example about S-duality of $\mathcal{N} = 4$ super Yang-Mills theories. After giving a brief review of S-duality and its action on dyonic charge sectors in section 4.5 we therefore show that the fusion rules obtained from the skeleton group commute with S-duality. In section 4.6 we come to a final piece of evidence for the relevance of the skeleton group which goes beyond the consistency checks of the preceding sections. For this purpose we introduce the skeleton gauge which is a minimal generalisation of 't Hooft's abelian gauge [29]. We argue that the skeleton group plays the role of an effective symmetry in the skeleton gauge. Moreover we proof that the skeleton gauge incorporates intrinsically non-abelian configurations, so-called Alice fluxes, which are excluded in the abelian gauge. Hence, compared to the abelian gauge, the skeleton gauge is particularly useful for exploring non-abelian phases of the theory which generalise Alice electrodynamics [30, 31, 32] and phases, as listed at the end of the section, that emerge from generalised Alice phases by condensation or confinement. The skeleton gauge is thus necessary to reveil certain phases of the theory which get "out of view" in the abelian gauge. An important example is a peculiar phase we find where particles have "lost" their charges.

4.1 LIE ALGEBRA CONVENTIONS

As explained in e.g. chaper 6 of [82], the Lie algebra $\mathfrak{g}$ of the gauge group G of rank r can be written as a sum of vector spaces

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha. \tag{4.1}$$

The Cartan subalgebra (CSA) $\mathfrak{g}_0$ is r-dimensional and from here on we shall denote it by $\mathfrak{t}$. The remaining one dimensional subspaces $\mathfrak{g}_\alpha$ are defined by:

$$[H, X] = \alpha(H)X \tag{4.2}$$

for all $H \in \mathfrak{t}$ and all $X \in \mathfrak{g}_\alpha$. In the Cartan-Weyl basis of $\mathfrak{g}$ we have:

$$[H_i, E_\alpha] = \alpha_i E_\alpha \qquad [E_\alpha, E_{-\alpha}] = \frac{2\alpha \cdot H}{\alpha^2} \tag{4.3}$$

where for each α the eigenvalue $\alpha_i := \alpha(H_i)$ is non-vanishing for at least one value of i. The r-dimensional vectors $\alpha = (\alpha_i)_{i=1,\ldots,r}$ are nothing but the roots of $\mathfrak{g}$ and each root α can thus thus be naturally interpreted as an element in $\mathfrak{t}^*$:

$$\alpha : \mathfrak{t} \to \mathbb{C}. \tag{4.4}$$

Intstead of the basis (H_i) for $\mathfrak{t}$ as used in equation (4.3) one can choose a basis for the CSA via

$$H_\alpha = 2\alpha^* \cdot H \tag{4.5}$$

where $\alpha^* = \alpha/\alpha^2$. We now find

$$[H_\alpha, E_\beta] = 2\alpha^* \cdot \beta E_\beta \qquad [E_\alpha, E_{-\alpha}] = H_\alpha. \tag{4.6}$$

The coroots H_α span the coroot lattice $\Lambda_{cr} \subset \mathfrak{t}$ and the roots span the root lattice $\Lambda_r \subset \mathfrak{t}^*$. The dual lattice of the coroot lattice is the weight lattice $\Lambda_w \subset \mathfrak{t}^*$ of $\mathfrak{g}$ generated by the fundamental weights. The dual lattice of the root lattice is the so-called magnetic weight lattice $\Lambda_{mw} \subset \mathfrak{t}$. Note that the weight lattice $\Lambda(G)$ of G satisfies $\Lambda_r \subset \Lambda(G) \subset \Lambda_w$, while the dual weight lattice $\Lambda^*(G)$ of G satisfies $\Lambda_{cr} \subset \Lambda^*(G) \subset \Lambda_{mw}$. It turns out [2] that $\Lambda^*(G)$ can be identified with the weight lattice $\Lambda(G^*)$ of GNO dual group G^*. The roots of G^* correspond to the coroot of G while the fundamental weights of G^* span Λ_{mw}.

4.2 CHARGE SECTORS OF THE THEORY

One of the key features of the skeleton group is that it reproduces the dyonic charge sectors of a Yang-Mills theory. To appreciate this one needs some basic understanding of the electric and magnetic charge lattices and the set of dyonic charge sectors.

4.2.1 ELECTRIC CHARGE LATTICES

To define the electric content of a gauge theory one starts by choosing an appropriate electric charge lattice Λ. Choosing an electric charge lattice corresponds to choosing a gauge group G such that Λ equals the weight lattice $\Lambda(G)$ of G. The electric charge lattice Λ can vary from the root lattice Λ_r to the weight lattice Λ_w of $\mathfrak{g}$. This corresponds to the fact that for a fixed the Lie algebra $\mathfrak{g}$ one can vary the Lie group G from $\overline{G}$ all the way to $\widetilde{G}$, where $\widetilde{G}$ is the universal covering group of G and $\overline{G}$ is the so-called adjoint group, which is the covering group divided by the center $Z(\widetilde{G})$. Note that the possible electric gauge groups are are not related as subgroups but rather by taking quotients.

4.2.2 MAGNETIC CHARGE LATTICES

Once the electric group G is chosen one is free to choose the magnetic content as long as the generalised Dirac quantisation condition [1, 2] is respected. The magnetic content is defined by fixing a magnetic charge lattice Λ^*. Just like on the electric side a choice for the magnetic charge lattice corresponds to a unique choice of a magnetic group G^* whose weight lattice $\Lambda(G^*)$ equals Λ^*. Again G^* can vary all the way from $\overline{G}^*$, the universal cover of G^*, to $\widetilde{G}^*$ which is the adjoint of G^*. This variation amounts to taking the magnetic charge lattice from the weight lattice Λ_{mw} to the root lattice Λ_{cr} of the fixed Lie algebra $\mathfrak{g}^*$ of G^*.
Even though G does neither completely fix G^* nor vice versa, the generalised quantisation condition as reviewed in section 2.3.1, does put some strong condition on the pair (G, G^*). First of all the Lie algebra $\mathfrak{g}$ of G uniquely fixes the Lie algebra $\mathfrak{g}^*$ of G^* and vice versa. Hence the universal covering groups $\widetilde{G}$ and $\overline{G}^*$ are uniquely related. Moreover, once G is fixed, the quantisation condition defines a maximal sublattice in Λ_{mw}, the weight lattice of $\overline{G}^*$. This special weight lattice equals the dual weight lattice $\Lambda^*(G)$ of G. Taking Λ^* equal to $\Lambda^*(G)$ amount to choosing G^* to be the GNO dual group of G. We thus see that once G is fixed G^* can vary between the adjoint group $\widetilde{G}^*$ and the GNO dual group of G. Analogously, if G^* is fixed G can vary between the GNO dual of G^* and the adjoint group $\overline{G}$ without violating the generalised Dirac quantisation condition.

Unless stated otherwise we shall take G and G^* to be their respective GNO duals. Note that if the fields present in the Lagrangian are only adjoint fields and one only wants to consider smooth monopoles it is natural to restrict G and G^* to be adjoint groups.

4.2.3 DYONIC CHARGE SECTORS

It was observed in [23, 24, 25, 26, 27, 28] that in a monopole background the global gauge symmetry is restricted to the centraliser C_g of the magnetic charge g. This implies that the charges of dyons are given by a pair (g, R_λ) where g is the usual magnetic charge corresponding to an element in the Lie algebra of G and R_λ is an irreducible representation of $C_g \subset G$. It is explained in [48] how these dyonic sectors can be relabelled in a convenient way. We shall give a brief review.

Since the magnetic charge is an element of the Lie algebra one can effectively view C_g as the residual gauge group that arises from adjoint symmetry breaking where the Lie algebra valued Higgs VEV is replaced by the magnetic charge. The Lie algebra of $\mathfrak{g}_g$ of C_g is easily determined. One can choose a gauge where the magnetic charge lies in the CSA of G. Note that this not fix g uniquely since the intersection of its gauge orbit and the CSA

corresponds to a complete Weyl orbit. Now since the generators H_α of the CSA commute one immediately finds that the complete CSA of G is contained in the Lie algebra of C_g. The remaining basis elements of $\mathfrak{g}_g$ are given by E_α whith α perpendicular to g. This follows from the fact that $[E_\alpha, H_\beta] = 2(\alpha \cdot \beta)/\beta^2 E_\alpha$. We thus see that the weight lattice of C_g is identical to the weight lattice of G, whereas the roots of C_g are a subset of the roots of G. Consequently the Weyl group $\mathcal{W}_g$ of C_g is the subgroup in the Weyl group $\mathcal{W}$ of G generated by the reflections in the hyperplanes perpendicular to the roots of C_g.
An irreducible representation R_λ of C_g is uniquely labelled by a Weyl orbit $[\lambda]$ in the weight lattice of C_g. Hence such a representation is fixed by a $\mathcal{W}_g$ orbit in the weight lattice of G. Now remember g itself is fixed up to Weyl transformations. Since $C_g \cong C_{w(g)}$ for all $w \in \mathcal{W}$ we find that (R_λ, g) is uniquely fixed by an equivalence class $[\lambda, g]$ under the diagonal action of W.

The ultimate goal of this paper is to find the fusion rules of dyons. We have explained that dyons are classified by an equivalence class of pairs $(\lambda, g) \in \Lambda(G) \times \Lambda(G^*)$ under the action of $\mathcal{W}$. By fusion rules we mean a set of rules of the form:

$$(R_{\lambda_1}, g_1) \otimes (R_{\lambda_2}, g_2) = \bigoplus_{[\lambda,g]} N^{\lambda,g}_{\lambda_1,\lambda_2,g_1,g_2} (R_\lambda, g) \tag{4.7}$$

where the coefficients $N^{\lambda,g}_{\lambda_1,\lambda_2,g_1,g_2}$ are positive or vanishing integers. Only for a finite number of terms these integers do not vanish. One may also expect that the product in equation (4.7) to be commutative and associative. Finally one should expect that the fusion rules of G and G^* are respected for at least the purely electric and respectively the purely magnetic cases.

4.3 Skeleton Group

In an abelian gauge theory with gauge group T the global electric symmetry is not restricted by any monopole background. The electric-magnetic symmetry is in that case thus described by $T \times T^*$. The global electric symmetry that can be realised in any monopole background is the maximal torus T generated by the CSA of G. The magnetic charges can be identified with representations of the dual torus T^*. Hence the electric-magnetic symmetry governing the gauge theory must contain $T \times T^*$. In the abelian case $T \times T^*$ is the complete symmetry group whereas in the non-abelian case there must exist some extension of $T \times T^*$ that respects the dyonic charge sectors. The simplest non-abelian extension of $T \times T^*$ respecting the dyonic charge sectors is the proto skeleton group $\mathcal{W} \ltimes (T \times T^*)$. Since the proto skeleton group contains the maximal tori of G and G^* while its product depends on the Weyl group action on these tori, its irreducible representations are labelled by the dyonic charge sectors of the Yang-Mills theory with gauge group G, as we shall explain in detail in section 4.4. However, the irreducible representations are only fixed once the so-called centraliser charges are given. It might be possible to assign some physical significance to these centraliser charges if the proto skeleton group is contained in the unified electric-magnetic symmetry of the theory. However, in that case one should expect that the electric part $\mathcal{W} \ltimes T$ is a subgroup of the electric group G.

Weyl group $\mathcal{W}^* = \mathcal{W}$	$\simeq \widetilde{W}^*/\widetilde{D}^*$	$\simeq$	W^*/D^*	$\simeq$	$\overline{W}^*/\overline{D}^*$
	$\uparrow$		$\uparrow$		$\uparrow$
Lift Weyl group	$\widetilde{W}^* \subset \widetilde{G}^*$	$\leftarrow$	$W^* \subset G^*$	$\leftarrow$	$\overline{W}^* \subset \overline{G}^*$
Dual torus	$\widetilde{T}^* = \mathbb{R}^r/\Lambda_w$	$\leftarrow$	$T^* = \mathbb{R}^r/\Lambda$	$\leftarrow$	$\overline{T}^* = \mathbb{R}^r/\Lambda_r$
	$\cap$		$\cap$		$\cap$
Dual group	$\widetilde{G}^* = \overline{G}^*/Z^*$	$\leftarrow$	G^*	$\leftarrow$	$\overline{G}^*$
	$\updownarrow$		$\updownarrow$		$\updownarrow$
Dual weight latt.	$\widetilde{\Lambda}^* = \Lambda_{cr}$	$\subset$	Λ^*	$\subset$	$\overline{\Lambda}^* = \Lambda_{cw}$
$\Uparrow$ **Magnetic** $\Uparrow$					
$- - - - - - - - -$	$\updownarrow$	$- - -$	$\updownarrow$	$- - -$	$\updownarrow$
$\Downarrow$ **Electric** $\Downarrow$					
Weigt lattice	$\widetilde{\Lambda} = \Lambda_w$	$\supset$	Λ	$\supset$	$\overline{\Lambda} = \Lambda_r$
	$\updownarrow$		$\updownarrow$		$\updownarrow$
Gauge Group	$\widetilde{G}$	$\rightarrow$	G	$\rightarrow$	$\overline{G} = \widetilde{G}/Z$
	$\cup$		$\cup$		$\cup$
Maximal torus	$\widetilde{T} = \mathbb{R}^r/\Lambda_{cr}$	$\rightarrow$	$T = \mathbb{R}^r/\Lambda^*$	$\rightarrow$	$\overline{T} = \mathbb{R}^r/\Lambda_{mw}$
Lift Weyl group	$\widetilde{W} \subset \widetilde{G}$	$\rightarrow$	$W \subset G$	$\rightarrow$	$\overline{W} \subset \overline{G}$
	$\downarrow$		$\downarrow$		$\downarrow$
Weyl group $\mathcal{W}$ $\simeq$	$\widetilde{W}/\widetilde{D}$	$\simeq$	W/D	$\simeq$	$\overline{W}/\overline{D}$

Usually this does not hold. In fact even the Weyl group $\mathcal{W}$ itself is usually not a subgroup of G. Nonetheless some important features of the proto skeleton group are shared with the skeleton group whose electric subgroup does indeed lie within G. We shall thus build up the construction of the skeleton group from the definition of the proto skeleton group and take advantage of the simplicity of the latter to explain some relevant properties of the skeleton group throughout the remainder of this paper.

4.3.1 Semi-direct products

The proto skeleton group and the skeleton group itself are both a semi-direct product. Here we shall recapitulate the definition of a semi-direct product.

One of several equivalent definitions is that if H is a subgroup of G and N a normal subgroup such that $G = NH$ and $H \cap N = \{e\}$ then G is a semi-direct product of H and N. Note that in contrast to the case with the direct product, a semi-direct product of two groups is in general not unique; if G and G' are both semi-direct products of H and N then it does not follow that G and G' are isomorphic. However, G is fixed up to isomorphism by the action of H on N. Let us denote $h \in H : n \in N \mapsto h \triangleright n = hnh^{-1} \in N$. Note that since N is a normal subgroup of G this action is well defined. Now define $H \ltimes N$ as the group whose elements are given by the set $H \times N$ and whose product is given by:

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 h_2 \triangleright n_2). \tag{4.8}$$

The inverse of (h, n) is given by $(h^{-1}, h^{-1} \triangleright n^{-1})$. $H \times \{e_N\}$ is a subgroup isomorphic to H while $\{e_H\} \times N$ is a normal subgroup isomorphic to N since

$$(h, e_N)(e_H, n)(h, e_N)^{-1} = (e_H, h \triangleright n). \tag{4.9}$$

The intersection $H \cap N$ equals $\{(e_H, e_n)\}$. Finally each element $(h, n) \in H \ltimes N$ satisfies

$$(h, n) = (e_H, n)(h, e_n) \in NH. \tag{4.10}$$

Hence, the full group $H \ltimes N$ is a semi-direct product of H and N in the sense above.

4.3.2 Maximal torus and its dual

The relevant normal group in the (proto) skeleton group is the product of the maximal torus of the gauge group and its dual. We shall describe these tori below.
Consider a particular gauge group G, and its Cartan subalgebra $\mathfrak{t}$. The weight lattice $\Lambda(G)$ of G is defined as follows: first define the lattice $\Lambda^*(G) \subset \mathfrak{t}$ as the kernel of the map

$$H \in \mathfrak{t} \mapsto \exp(2\pi i H) \in G. \tag{4.11}$$

The weight lattice $\Lambda(G) \subset \mathfrak{t}^*$ is the dual lattice of $\Lambda^*(G)$. Also note that

$$\Lambda_r \subset \Lambda(G) \subset \Lambda_w \subset \mathfrak{t}^* \tag{4.12}$$

and

$$\Lambda_{cr} \subset \Lambda^*(G) \subset \Lambda_{mw} \subset \mathfrak{t}. \tag{4.13}$$

So far, all vector spaces where considered over $\mathbb{C}$. However, by declaring the basis $\{H_\alpha\}$ of $\mathfrak{t}$ real and considering its real span we obtain a real vector space $\mathfrak{t}_\mathbb{R}$. Similarly, the real span of the roots α is a real vector space, denoted by $\mathfrak{t}^*_\mathbb{R}$ in the following. We then define the maximal torus of G via:

$$T = \mathfrak{t}_\mathbb{R}/(2\pi\Lambda^*(G)) \tag{4.14}$$

and the dual torus via

$$T^* = \mathfrak{t}^*_\mathbb{R}/(2\pi\Lambda(G)). \tag{4.15}$$

A convenient way to represent T is as follows. Let $\widetilde{G}$ be the universal cover of G. The dual weight lattice $\Lambda^*(\widetilde{G})$ for $\widetilde{G}$ equals the coroot lattice $\Lambda_{cr}(\widetilde{G})$. The generators of this lattice are the coroots H_{α_i} with α_i a simple root. One thus finds that $T_{\widetilde{G}}$ is explicitly parametrised by $H = \sum_{i=1}^r \theta_i H_{\alpha_i}$ where $r = \text{rank}(\mathfrak{g})$ with $\theta_i \in [0, 2\pi)$. Note that the image of $T_{\widetilde{G}}$ in $\widetilde{G}$ defined by $H \mapsto \exp(2\pi i H)$ is isomorphic to $T_{\widetilde{G}}$. The set of irreducible representations of the abelian group $T_{\widetilde{G}} \subset \widetilde{G}$ is given by the weight lattice Λ_w of $\widetilde{G}$.
Let us return to G which is isomorphic to $\widetilde{G}/Z$, where Z is the group $\Lambda^*(G)/\Lambda_{cr} \subset T_{\widetilde{G}}$. Similarly $T_G = T_{\widetilde{G}}/Z$. The irreducible representations of T_G are given by those of $T_{\widetilde{G}}$ which represent Z trivially. This set of representations corresponds to $\Lambda(G)$.

The dual torus T^* is isomorphic to the maximal torus of the GNO dual group G^* of G. The roots of G^* are precisely the coroots α^* of G and the coroots H_{α^*} of G^* correspond to the roots α of G. Using analogous arguments we thus find that any element T^* can be uniquely represented as $H^* = \sum_{i=1}^r \theta_i^* H_{\alpha_i^*}$ up to an element in $Z^* = \Lambda(G)/\Lambda_r$. Note that the irreducible representations of T^* correspond to the set $\Lambda^*(G) = \Lambda(G^*)$.
If we take the magnetic group G^* not equal to the GNO dual of G but instead to some quotient then we define T^* to be the maximal torus of G^*. T^* is then found from the maximal torus of the GNO dual group by using the same discrete group that relates G^* itself to the GNO dual group of G.

4.3.3 WEYL GROUP ACTION

The semi-direct product that plays a role in the definition of the (proto) skeleton group is defined with respect to the action of the Weyl group on the maximal torus of G and its dual torus. We shall briefly discuss this action.

The Weyl group is a subgroup of the automorphism group of the root system generated by the Weyl reflections

$$w_\alpha : \beta \mapsto \beta - \frac{2\alpha \cdot \beta}{\alpha^2}\alpha. \tag{4.16}$$

By linearity the action of the Weyl group can be extended to the whole root lattice, the weight lattice and $\mathfrak{t}^*$.

$$w_\alpha : \lambda \mapsto \lambda - \frac{2\alpha \cdot \lambda}{\alpha^2}\alpha. \tag{4.17}$$

Note that w_α simply corresponds to the reflection in the hyperplane in $\mathfrak{t}^*$ orthonormal to the root α.

Remember that $\mathfrak{t}^*$ is the dual space of $\mathfrak{t}$, the CSA of G. The action of $w \in \mathcal{W}$ on $H \in \mathfrak{t}$ is defined by $\alpha(w(H)) = w^{-1}(\alpha)(H)$. From this relation one finds

$$w_\alpha(H) = H - \frac{2\langle H, H_\alpha \rangle}{\langle H_\alpha, H_\alpha \rangle} H_\alpha. \tag{4.18}$$

In particular one finds

$$w_\alpha(\beta^*) = \beta^* - \frac{2\beta^* \cdot \alpha^*}{(\alpha^*)^2}\alpha^* \tag{4.19}$$

and

$$w^{-1}(H_\alpha) = w^{-1}(2\alpha^* \cdot H) = 2w(\alpha^*) \cdot H = H_{w(\alpha)}. \tag{4.20}$$

Remember that the maximal torus of G is, up to discrete identifications, isomorphic to $U(1)^r$ where each $U(1)$ factor is generated by one H_α. Consequently there is a natural action of the Weyl group on the maximal torus given by

$$w \in \mathcal{W} : \exp\left(i\theta_i H_{\alpha_i}\right) \in T \mapsto \exp\left(i\theta_i H_{w(\alpha_i)}\right) \in T. \tag{4.21}$$

Analogously one can define the action of the Weyl group on the dual torus

$$w \in \mathcal{W} : \exp\left(i\theta_i^* H_{\alpha_i^*}\right) \in T^* \mapsto \exp\left(i\theta_i^* H_{w(\alpha_i^*)}\right) \in T^*. \tag{4.22}$$

4.3.4 PROTO SKELETON GROUP

The proto skeleton group associated to G is defined as

$$\mathcal{W} \ltimes (T \times T^*), \tag{4.23}$$

where $\mathcal{W}$ is the Weyl group of G. T and T^* are the maximal torus of G and G^* as discussed in section 4.3.2. The semi-direct product which appears in the definiion of the proto skeleton group is given by:

$$w \in \mathcal{W} : (t, t^*) \in T \times T^* \mapsto (w(t), w(t^*)) \in T \times T^* \tag{4.24}$$

where $w(t)$ and $w(t^*)$ are defined as in equation (4.21) and (4.22). Equivalently, the proto skeleton group equals $(\mathcal{W} \ltimes T) \ltimes T^*$ if one defines the action of $\mathcal{W} \ltimes T$ on T^* by $(w,t) \triangleright t^* \mapsto w(t^*)$, where the action of $\mathcal{W}$ on T^* is again defined as above. Note that the definition of the proto skeleton group is completely symmetric with respect to the interchange of electric and magnetic groups.

4.3.5 Definition of the skeleton group

The proto skeleton group contains the maximal tori of G and G^* while the group product depends on the Weyl group action on these tori. Hence, its irreducible representations are labelled by the dyonic charge sectors of the Yang-Mills theory with gauge group G, as we shall explain in detail in section 4.4. However, the irreducible representations are only fixed up to so-called centraliser charges. It might be possible to assign some physical significance to these centraliser charges if the proto skeleton group could be argued to be a subgroup of the full symmetry of the theory. However, in that case one should expect that the electric part $\mathcal{W} \ltimes T$ is a subgroup of the electric group G. We might also require that the magnetic part $\mathcal{W} \ltimes T^*$ to be a subgroup of the magnetic group G^*. Usually neither of these requirements is fulfilled. In fact even the Weyl group $\mathcal{W}$ itself is usually not a subgroup of G or G^*. However, recall that the Weyl group is isomorphic to the normaliser of T in G moduli the centraliser of T. This implies that the Weyl group can be lifted to G and, since the Weyl group only depends on the Lie algebra, it can actually be lifted to any Lie group with this fixed algebra. We will use this fact to show that we can define the skeleton group such that its electric part is indeed a subgroup of G.

According to [83], a natural finite lift $\overline{W}$ of $\mathcal{W}$ into the group of automorphisms of $\mathfrak{g}$ is defined as follows. For any simple root α of G, we define a lift $\overline{w}_\alpha$ of the Weyl reflection w_α by

$$\overline{w}_\alpha = \mathrm{Ad}(x_\alpha) \tag{4.25}$$

with

$$x_\alpha = \exp\left(\frac{i\pi}{2}(E_\alpha + E_{-\alpha})\right). \tag{4.26}$$

The $\overline{w}_\alpha$ generate $\overline{W}$, which is a finite subgroup of the automorphism group of $\mathfrak{g}$. Note that $\overline{W}$ is a also a subgroup of the adjoint group $\overline{G}$ of G. $\overline{W}$ has an abelian normal subgroup $\overline{D}$ generated by the elements $\overline{w}_\alpha^2$ and we have $\mathcal{W} = \overline{W}/\overline{D}$.
If one wants to lift $\mathcal{W}$ into the group G itself, rather than into its adjoint representation, one can do this by lifting $\overline{W} \subset G/Z_G$ into G. Such a lift W' of $\mathcal{W}$ can be defined as the preimage of $\overline{W}$ under the projection from G to its adjoint group G/Z_G. Alternatively, one can define a lift W of $\mathcal{W}$ into G as the group generated by the elements x_α of G. In general, we might have $W \neq W'$, although it is clear that $W \subset W'$. In the remainder of this paper we shall ignore this possible subtlety and only consider the lift W. We shall

also use the abelian normal subgroup $D \subset W$ defined by $D = W \cap T$.

We now introduce the skeleton group S as

$$S = W \ltimes (T \times T^*)/D, \tag{4.27}$$

where the action of $d \in D$ is by simultaneous left multiplication on $W \ltimes T$. The action of W on the two maximal tori is the usual conjugation action and it factors over the quotient $\mathcal{W}$ of W, i.e. every element $w \in W$ acts just like the corresponding element of the Weyl group $\mathcal{W}$. Note that equivalently we can write:

$$S = \frac{W \ltimes T}{D} \ltimes T^*. \tag{4.28}$$

We define the electric subgroup S_{el} of S as

$$S_{el} = \{s \in S \mid s = (w, t, 1)D,\ w \in W,\ t \in T\}. \tag{4.29}$$

One may now define $\phi : W \ltimes T \to G$ by

$$\phi(w, t) = wt^{-1}. \tag{4.30}$$

It is easy to check that ϕ is a homomorphism into $N_T \subset G$, the normaliser of T. The kernel of ϕ is precisely the set of elements $(d, d) \in W \ltimes T$, with necessarily $d \in D$. As a result, S_{el} is isomorphic to the image of ϕ, which is in turn a subgroup of $N_T \subset G$ and we have achieved our goal to make the electric part of the skeleton group a subgroup of the electric group.
With the definition above one should not expect the magnetic subgroup S_{mag} defined as

$$S_{mag} = \{s \in S \mid s = (w, 1, t^*)D,\ w \in W,\ t^* \in T^*\} \tag{4.31}$$

to be a subgroup of G^* since $S_{mag} = \mathcal{W} \ltimes T^*$ and the Weyl group $\mathcal{W}$ of G and G^* is in general not a subgroup of G^*. However, one can introduce the dual group S^* and define it to be the skeleton group of G^*. The electric subgroup S^*_{el} is then of course a subgroup of G^*.

4.4 Representation theory of the skeleton group

In this section we discuss some general properties of the representations of the (proto) skeleton group and its fusion rules. We focus in particular on $SU(2)$ as an example. Further detailed examples are worked out in appendix C and D.

4.4.1 Representation theory for semi-direct products

The classification of the irreducible representations of a semi-direct product involving an abelian normal subgroup is well known, see e.g. [84]. A (finite dimensional) irreducible representation of $H \ltimes N$, where N is abelian, is up to unitary equivalence determined by an irreducible representation of N and a so-called centraliser representation. This centraliser representation is defined as follows: since N is abelian each of its irreducible representations is given by a function $\lambda : N \to \mathbb{C}$ that respects the group product, i.e. each irreducible representation is a character of N. The action of H on N with respect to which the semi-direct product is defined can be lifted to an action on the characters:

$$h \in H : \lambda \mapsto h(\lambda) \tag{4.32}$$

where

$$h(\lambda) : n \in N \mapsto \lambda(h^{-1} \triangleright n) \in \mathbb{C}. \tag{4.33}$$

Let $H_\lambda \subset H$ be the centraliser group of λ such that for all $h \in H_\lambda$ $h(\lambda) = \lambda$. For any irreducible representation γ of H_λ one can easily check that

$$\Pi^\lambda_\gamma : (h, n) \mapsto \lambda(n)\gamma(h) \tag{4.34}$$

defines an irreducible representation of $H_\lambda \ltimes N$.

A representation of $H \ltimes N$ can now be constructed as an induced representation, see e.g. section 3.3 of [50]. For a general group G one starts with a matrix representation γ of a subgroup $H \subset G$ acting on a vector space V. We shall assume that G/H is a finite set. One can now construct a unique finite dimensional representation $\Pi^{G/H}_\gamma$ of G by introducing the vector space $\oplus_{\mu \in G/H} V_\mu$, where V_μ is a copy of V for each coset $\mu \in G/H$. While H simply acts on each copy V_μ simultaneously, all elements g outside H also mix up these copies by their action on the cosets.
By choosing a representative g_μ for each coset μ one can construct the induced representation explicitly. Given g in G one has $gg_\mu = g_\nu h$ for some coset ν and some $h \in H$. Hence g maps V_μ to V_ν with an additional action of h. Consequently the matrix corresponding to g only has diagonal elements if for some coset $g(\sigma) = \sigma$, i.e. $g^{-1}_\sigma g g_\sigma \in H$. Therefore the character χ_G of the induced representation can be computed from the character χ_H of the representation one started out with:

$$\chi_G(g) = \sum_{g(\sigma)=\sigma} \chi_H(g^{-1}_\sigma g g_\sigma). \tag{4.35}$$

As a first step we can use the method described above to construct the induced representation of the centraliser representation γ. Note that H/H_λ is precisely the H-orbit $[\lambda]$ in the set of characters of N. Let V be the vector space on which the centraliser representation

γ of H_λ acts. We denote the elements of V_μ by $|\,\mu, v\,\rangle$ where we understand $\mu \in [\lambda]$ and $v \in V$. For each coset in H/H_λ and hence for each $\mu \in [\lambda]$ we choose a representative $h_\mu \in H$ such that $h_\mu(\lambda) = \mu$. Given $h \in H$ we have $hh_\mu = h_\nu h'$ with $\nu = h(\mu) \in [\lambda]$ and $h' \in H_\lambda$. We can now define the induced representation $\Pi_\gamma^{[\lambda]}$ for H by

$$\Pi_\gamma^{[\lambda]}(h)|\,\mu, v\,\rangle = |\,\nu, \gamma(h_\nu^{-1} h h_\mu) v\,\rangle. \tag{4.36}$$

A representation of $H \ltimes N$ is found just as easily. Note that $H \ltimes N/H_\lambda \ltimes N = H/H_\lambda = [\lambda]$. This implies that one can choose the representants of the cosets in the semi-direct product to be of the form (h_μ, e) with $\mu \in [\lambda]$. From the semi-direct product (4.8) we now find

$$(h, n)(h_\mu, e) = (h_\nu h', n) = (h_\nu, e)(h', h_\nu^{-1} \triangleright n) \tag{4.37}$$

where $h' = h_\nu^{-1} h h_\mu$. We thus find and induced representation $\Pi_\gamma^{[\lambda]}$ of Π_γ^λ defined by

$$\Pi_\gamma^{[\lambda]}(h, n)|\,\mu, v\,\rangle = |\,\nu, \Pi_\gamma^\lambda(h_\nu^{-1} h h_\mu, h_\nu^{-1} \triangleright n) v\,\rangle = \nu(n)|\,\nu, \gamma(h_\nu^{-1} h h_\mu) v\,\rangle. \tag{4.38}$$

Note that we used equation (4.33) to rewrite $\lambda(h_\nu^{-1} n) = h_\nu(\lambda)(n) = \nu(n)$. We thus see that each V_μ defines a representation of N since

$$\Pi_\gamma^{[\lambda]}(e, n)|\,\mu, v\,\rangle = \mu(n)|\,\mu, v\,\rangle. \tag{4.39}$$

It is easily seen that $\oplus_{\mu \in [\lambda]} V_\mu$ does not contain any invariant subspaces and thus that $\Pi_\gamma^{[\lambda]}$ is an irreducible representation of $H \ltimes N$. Equation (4.38) clearly shows that this representation only depends on the orbit $[\lambda]$ and the centraliser representation γ but not on the initial choice for the representant λ of $[\lambda]$. One can also check that up to unitary transformations $\Pi_\gamma^{[\lambda]}$ does not depend on the choice of representants for the elements in H/H_λ. Finally, it turns out that all irreducible representations of $H \ltimes N$ can be obtained in this way.

The decomposition of a tensor product representation into irreducible representations can be computed with the inner product of the characters. The characters for an irreducible representations $\Pi_\gamma^{[\lambda]}$ of $H \ltimes N$ can be found from equation (4.35):

$$\begin{aligned} \chi_\gamma^{[\lambda]}(h, n) &= \sum_{\mu \in [\lambda]} \delta_{h(\mu),\mu} \chi_\gamma^\lambda((h_\mu^{-1}, e)(h, n)(h_\mu, n)) \\ &= \sum_{\mu \in [\lambda]} \delta_{h(\mu),\mu} \chi_\gamma^\lambda(h_\mu^{-1} h h_\mu, h_\mu^{-1} n) \\ &= \sum_{\mu \in [\lambda]} \delta_{h(\mu),\mu} \chi_\gamma(h_\mu^{-1} h h_\mu) \lambda(h_\mu^{-1} n) \\ &= \sum_{\mu \in [\lambda]} \delta_{h(\mu),\mu} \chi_\gamma(h_\mu^{-1} h h_\mu) \mu(n). \end{aligned} \tag{4.40}$$

One can check that these characters are orthogonal:

$$\begin{aligned}
\left\langle \chi_\gamma^{[\rho]}, \chi_\alpha^{[\sigma]} \right\rangle &= \int_{H\times N} \chi_\gamma^{[\rho]}(h,n)\chi^{*[\sigma]}_{\ \alpha}(h,n)dhdn \\
&\int_{H\times N} \sum_{\mu\in[\rho]} \delta_{h(\mu),\mu}\chi_\gamma(h_\mu^{-1}hh_\mu)\mu(n) \sum_{\nu\in[\sigma]} \delta_{h(\nu),\nu}\chi^*_\alpha(h_\nu^{-1}hh_\nu)\nu^*(n)dhdn \\
&= \sum_{\mu\in[\rho]}\sum_{\nu\in[\sigma]} \int_N \mu(n)\nu^*(n)dn \int_H \delta_{h(\mu),\mu}\delta_{h(\nu),\nu}\chi_\gamma(h_\mu^{-1}hh_\mu)\chi^*_\alpha(h_\nu^{-1}hh_\nu)dh \\
&= \sum_{\mu\in[\rho]}\sum_{\nu\in[\sigma]} \delta_{\mu\nu} \int_N dn \int_H \delta_{h(\mu),\mu}\chi_\gamma(h_\mu^{-1}hh_\mu)\chi^*_\alpha(h_\mu^{-1}hh_\mu)dh \qquad (4.41)\\
&= \delta_{[\rho][\sigma]} \sum_{\mu\in[\rho]} \int_N dn \int_{H_\rho} \chi_\gamma(h')\chi^*_\alpha(h')dh \\
&= \delta_{[\rho][\sigma]}\delta_{\gamma\alpha} \sum_{\mu\in[\rho]} \int_{H_\rho\times N} dhdn \\
&= \delta_{[\rho][\sigma]}\delta_{\gamma\alpha}\, \mathrm{dim}(H\times N).
\end{aligned}$$

For $a = \Pi_\alpha^{[\sigma]}$, $b = \Pi_\beta^{[\eta]}$ and $c = \Pi_\gamma^{[\rho]}$ we find:

$$\begin{aligned}
\langle \chi_c, \chi_{a\otimes b}\rangle &= \int_{H\times N} \chi_c(h,n))\chi^*_a(h,n)\chi^*_b(h,n)dhdn \\
&= \int_{H\times N} \sum_{\mu\in[\rho]} \delta_{h(\mu),\mu}\chi_\gamma(h_\mu^{-1}hh_\mu)\mu(n)\times \\
&\quad \sum_{\nu\in[\sigma]} \delta_{h(\nu),\nu}\chi^*_\alpha(h_\nu^{-1}hh_\nu)\nu^*(n) \sum_{\zeta\in[\eta]} \delta_{h(\zeta),\zeta}\chi^*_\beta(h_\zeta^{-1}hh_\zeta)\zeta^*(n)dhdn \\
&= \sum_{\mu\in[\rho]}\sum_{\nu\in[\sigma]}\sum_{\zeta\in[\eta]} \int_N \mu(n)\nu^*(n)\zeta^*(n)dn \int_H \delta_{h(\mu),\mu}\delta_{h(\nu),\nu}\delta_{h(\zeta),\zeta}\times \\
&\quad \chi_\gamma(h_\mu^{-1}hh_\mu)\chi^*_\alpha(h_\nu^{-1}hh_\nu)\chi^*_\beta(h_\zeta^{-1}hh_\zeta)dh \\
&= \sum_{\mu\in[\rho]}\sum_{\nu\in[\sigma]}\sum_{\zeta\in[\eta]} \delta_{\mu,\nu\zeta} \int_{H\times N} \delta_{h(\mu),\mu}\delta_{h(\nu),\nu}\delta_{h(\zeta),\zeta}\times \qquad (4.42)\\
&\quad \chi_\gamma(h_\mu^{-1}hh_\mu)\chi^*_\alpha(h_\nu^{-1}hh_\nu)\chi^*_\beta(h_\zeta^{-1}hh_\zeta)dhdn.
\end{aligned}$$

We thus see that the fusion rules of $H \ltimes N$ can be expressed in terms of the multiplication in the character group of N and integrals involving the characters of the centraliser representations.

4.4.2 Weyl orbits and centraliser representations

Since the maximal tori of G and G^* are abelian and the Weyl group of G is a finite group, the results of section 4.4.1 can be applied directly to the proto skeleton group. Below we shall work out some general properties of its irreducible representations and prove our claim that these representations reproduce the charge sectors of the gauge theory with gauge group G. For explicit examples we refer to section C.

An irreducible representation of $\mathcal{W} \ltimes (T \times T^*)$ is labelled by an orbit in character group of $T \times T^*$ and by a centraliser representation. The character group, i.e. the set of irreducible representations of $T \times T^*$ is precisely given by $\Lambda(G) \times \Lambda(G^*)$ as follows from the discussion in section 4.3.2. The diagonal action of the Weyl group defining the semi-direct product of the proto skeleton group induces a diagonal action in the character group:

$$w \in \mathcal{W} : (\lambda, g) \in \Lambda(G) \times \Lambda^*(G) \mapsto (w(\lambda), w(g)) \in \Lambda(G) \times \Lambda^*(G) \tag{4.43}$$

where

$$\begin{aligned} (w(\lambda), w(g)) : (t, t^*) \in T \times T^* &\quad \to \quad \lambda(w^{-1}(t))g(w^{-1}(t^*)) \in \mathbb{C} \\ \big(\exp(i\theta_i H_{\alpha_i}), \exp(i\theta_i^* H_{\alpha_i^*})\big) &\quad \mapsto \quad \exp(i\theta_i 2w(\lambda) \cdot \alpha_i^* + i\theta_i^* 2w(g) \cdot \alpha_i). \end{aligned} \tag{4.44}$$

Here we used equations (4.21) and (4.22) together with

$$H_{w(\alpha)}|\,\lambda\,\rangle = 2\lambda \cdot w(\alpha^*)|\,\lambda\,\rangle = 2w^{-1}(\lambda) \cdot \alpha^*|\,\lambda\,\rangle, \tag{4.45}$$

and similarly

$$H_{w(\alpha^*)}|\,g\,\rangle = 2g \cdot w(\alpha)|\,g\,\rangle = 2w^{-1}(g) \cdot \alpha|\,g\,\rangle. \tag{4.46}$$

We thus see that an irreducible representation of the proto skeleton group carries a label that corresponds to an orbit $[\lambda, g]$ in $\Lambda(G) \times \Lambda(G^*)$. These labels are precisely the dyonic charge sectors of Kapustin [48] as discussed in section 4.2.3.

Let us discuss the centraliser representations in some more detail. These representations are irreducible representations of the centraliser $\mathcal{W}_{(\lambda,g)} \subset W$ of the dyonic charge (λ, g). Since $w(\lambda, g)$ is $(w(\lambda), w(g))$ we see that $\mathcal{W}_{(\lambda,g)} = \mathcal{W}_\lambda \cap \mathcal{W}_g \subset \mathcal{W}_g$. $\mathcal{W}_g$ is the Weyl group of the centraliser group $C_g \subset G$ introduced in section 4.2.3. Similarly $\mathcal{W}_{(\lambda,g)}$ is the Weyl group of some Lie group $C_{(\lambda,g)} \subset C_g$. Since the dyonic charge sector $[\lambda, g]$ corresponds to a unique pair (R_λ, g), where R_λ is an irreducible representation of C_g, one would now expect that the allowed centraliser representations for $\mathcal{W}_{(\lambda,g)} \subset \mathcal{W} \ltimes (T \times T^*)$ fit into an irreducible representation R_λ of $C_g \subset G$. Unfortunately, such a relation between representations is in principle absent since in general $\mathcal{W}$ is not a subgroup of G and $\mathcal{W}_g$ is not a subgroup of C_g. However, the skeleton group is constructed in such a way that this relation with the centraliser group in G can be established, as we shall discuss in the next section.

4.4.3 Representations of the skeleton group

Important features of the skeleton group S are that it is an extension of $T \times T^*$ respecting the dyonic charge sectors and moreover that its electric factor S_{el} is a subgroup of G. This implies that representations of G decompose into irreducible representations of the skeleton group with trivial magnetic charges. In this section we describe the representations of the skeleton group in general terms and clarify the relation with G-representations. The $SU(2)$-case is worked out explicitly in section 4.4.5. More examples can be found in appendix D.

The representations of S correspond precisely to the representations of $W \ltimes (T \times T^*)$ whose kernel contain the normal subgroup D. Since $W \ltimes (T \times T^*)$ is a semidirect product and the lift W acts in the same way on $T \times T^*$ as the Weyl group W, its irreducible representations are labelled by a $\mathcal{W}$-orbit in the weight lattice of $T \times T^*$ and by an irreducible representation of the centraliser in W of this orbit. Explicitely, let $[\lambda, g]$ denote the $\mathcal{W}$-orbit containing (λ, g) and let γ denote an irreducible representation of the centraliser $N_{(\lambda,g)} \subset W$ of (λ, g). Now for any $(\mu, h) \in [\lambda, g]$, choose some $x_{(\mu,h)} \in W'$ such that $x_{(\mu,h)}(\lambda, g) = (\mu, h)$ and define $V_\gamma^{[\lambda,g]}$ to be the vector space spanned by $\{|\mu, h, e_i^\gamma\rangle\}$, where $\{e_i^\gamma\}$ is a basis for the vector space V_γ on which γ acts. If we apply equation (4.38) for the induced representation of a semi-direct product we find that the irreducible representation $\Pi_\gamma^{[\lambda,g]}$ of $W \ltimes (T \times T^*)$ acts on $V_\gamma^{[\lambda,g]}$ as follows:

$$\Pi_\gamma^{[\lambda,g]}(w, t, t^*)|\, \mu, h,\, v\,\rangle = w(\mu)(t)w(h)(t^*)|\, w(\mu), \gamma(x_{w(\mu)}^{-1} w x_\mu)v\,\rangle. \qquad (4.47)$$

The irreducible representations of $W \ltimes (T \times T^*)$ with trivial centraliser labels are in one-to-one relation with the electric-magnetic charge sectors, just as is the case for the irreducible representations of the proto skeleton group. However, in general not all of these representations are representations of S. The allowed representations satisfy

$$\Pi_\gamma^{[\lambda,g]}(d, d, 1)|\, \mu, h,\, v\,\rangle = |\, \mu, h,\, v\,\rangle, \qquad (4.48)$$

which implies

$$d(\mu)(d)d(h)(1)|\, d(\mu), d(h), \gamma(x_{d(\mu)}^{-1} d x_\mu)v\,\rangle = |\, \mu, h,\, v\,\rangle. \qquad (4.49)$$

Since $d \in T$ we have $d(\mu) = \mu$ and we find that $\Pi_\gamma^{[\lambda,g]}$ is a representation of S if

$$\mu(d)|\, \mu, h, \gamma(x_\mu^{-1} d x_\mu)v\,\rangle = |\, \mu, h,\, v\,\rangle \qquad \forall\, |\, \mu, h,\, v\,\rangle \in V_\gamma^{[\lambda,g]}. \qquad (4.50)$$

This condition is satisfied if D acts trivially on all vectors of the form $|\, \lambda, v\,\rangle$. To show this we note that $\mu(t) = x_\mu(\lambda)(t) = \lambda(x_\mu^{-1} \triangleright t) = \lambda(x_\mu^{-1} t x_\mu)$. As mentioned in section 4.3.5 D is a normal subgroup of W, i.e. $x_\mu^{-1} d x_\mu = d' \in D$. Hence for the action of D on $|\, \mu, v\,\rangle$ we thus have

$$\mu(d)|\, \mu, \gamma(x_\mu^{-1} d x_m u)v\,\rangle = \lambda(d')|\, \mu, \gamma(d')v\,\rangle = |\, \mu\,\rangle \lambda(d')\gamma(d')|\, v\,\rangle. \qquad (4.51)$$

Now if

$$\lambda(d)|\,\lambda,\gamma(d)v\,\rangle = |\,\lambda\,\rangle\lambda(d)\gamma(d)|\,v\,\rangle = |\,\lambda\,\rangle|\,v\,\rangle \tag{4.52}$$

for all $d \in D$ we find that D acts trivially on $V_{\gamma}^{[\lambda,g]}$. The question that remains is if there always exists centraliser representation γ of $W_{(\lambda,g)} \subset W$ that satisfies this constaint.
Note that equation (4.52) is precisely the constraint one would obtain for representations of the electric part $(W \ltimes T)/D$ of the skeleton group except that γ would be an irreducible representation of a possible larger subgroup $W_\lambda \subset W$, i.e. $W_{(\lambda,g)} \subset W_\lambda$. This means however that the restriction $\gamma_{|W_{(\lambda,g)}}$ of an allowed electric centraliser representation γ of W_λ automatically satisfies (4.52). Consequently there exists an irreducible represention of S for a given orbit $[\lambda, g]$ if there exists an irreducible representation of S_{el} for a given orbit $[\lambda]$.
It is easily seen that an irreducible representation of S_{el} labelled by $[\lambda]$ exists if λ lies in the weight lattice of G. As proven in section 4.3.5 S_{el} is a subgroup of G and thus all representations of the gauge group fall apart into representations of S_{el}. Moreover both the gauge group and the skeleton group contain the maximal torus T. Hence all representations of T that appear in the restriction of G-representations must also appear in the restriction of a representation of S_{el} to T. From the representation theory of G we know that all irreducible representations of T come up in this way and hence all Weyl orbits in the weight lattice of G give rise to a representation of the skeleton group. We finally note that an irreducible representation of G with highest weight λ leads to a representation of S_{el} which has a one-dimensional centraliser representation. If a represention of G has a weight with multiplicity greater than one it may give rise to an allowed centraliser representation acting on a space which has more dimensions.

4.4.4 Fusion rules

Here we discuss some general properties of the fusion rules of the (proto) skeleton group. We shall restrict our discussion to the electric-magnetic charges and ignore the centraliser representation for the most part. The fusion rules of the dyonic charges are found by combining the Weyl orbits of the representations. An elegant way do deal with this combinatorics is the use a group ring.
Below we define a homomorphism, denoted by "Char" from the representation ring of the (proto) skeleton group to the Weyl invariant part $\mathbb{Z}[\Lambda]^{\mathcal{W}}$ of the group ring $\mathbb{Z}[\Lambda \times \Lambda^*]$ where $\Lambda \times \Lambda^*$ is the weight lattice of $T \times T^*$. This group ring has an additive basis given by the elements $e_{(\lambda,g)}$ with $(\lambda, g) \in \Lambda \times \Lambda^*$. The multiplication of the group ring is defined by $e_{(\lambda_1,g_1)}e_{(\lambda_2,g_2)} = e_{(\lambda_1+\lambda_2,g_1+g_2)}$. Finally the action of the Weyl group on the weight lattice induces an action on the group ring given by

$$w \in \mathcal{W} : e_{(\lambda,g)} \mapsto e_{w(\lambda),w(g)}. \tag{4.53}$$

A natural basis for the ring $\mathbb{Z}[\Lambda \times \Lambda^*]^{\mathcal{W}}$ is the set of elements of the form

$$e_{[\lambda,g]} := \sum_{(\mu,h)\in[\lambda,g]} e_{(\mu,h)}, \tag{4.54}$$

where $[\lambda, g]$ is a Weyl orbit in the weight lattice.

The homomorphism Char of the representation ring of the (proto) skeleton group to $\mathbb{Z}[\Lambda\times\Lambda^*]$ is defined by mapping $|\,\mu, h, v\,\rangle \in V_\gamma^{[\lambda,g]}$ to $e_{(\mu,h)} \in \mathbb{Z}[\Lambda \times \Lambda^*]$. Consequently for an irreducible representation $\Pi_\gamma^{[\lambda,g]}$ of the (proto) skeleton group we have $e_{[\lambda,g]}$ in $\mathbb{Z}[\Lambda \times \Lambda^*]^{\mathcal{W}}$

$$\text{Char} : \Pi_\gamma^{[\lambda,g]} \mapsto \text{dim}(V_\gamma)e_{[\lambda,g]}. \tag{4.55}$$

Note that if γ is a trivial centraliser representation or at least 1-dimensional then Char maps to a basis element of the group algebra.

Char respects the addition and multiplication in the representation ring since

$$\text{Char} \quad : \Pi_{\gamma_1}^{[\lambda_1,g_1]} \oplus \Pi_{\gamma_2}^{[\lambda_2,g_2]} \mapsto \text{dim}(V_{\gamma_1})e_{[\lambda_1,g_1]} + \text{dim}(V_{\gamma_2})e_{[\lambda_2,g_2]} \tag{4.56}$$

$$\text{Char} \quad : \Pi_{\gamma_1}^{[\lambda_1,g_1]} \otimes \Pi_{\gamma_2}^{[\lambda_2,g_2]} \mapsto \text{dim}(V_{\gamma_1})\text{dim}(V_{\gamma_2})e_{[\lambda_1,g_1]}e_{[\lambda_2,g_2]}. \tag{4.57}$$

We can use this to retrieve the fusion rules for the dyonic charge sectors since the expansion of skeleton group representations in irreducible representations corresponds to expanding products in the Weyl invariant group ring into basis elements:

$$e_{[\lambda_1,g_1]}e_{[\lambda_2,g_2]} = \sum_{[\lambda,g]} N^{\lambda,g}_{\lambda_1,\lambda_2,g_1,g_2} e_{[\lambda,g]}. \tag{4.58}$$

If one restricts to the purely electric sector, i.e. $g = 0$, such that the centraliser $C_g \subset G$ equals G itself, one should expect to retrieve the fusion rules of G. As was noticed by Kapustin in [81] equation (4.58) does not correspond to the decoposition of tensor products of G representations. However, the fusion rules of the (proto) skeleton group does also involve the centraliser representations. In particular the dimensions of the centraliser representations satisfy

$$\Pi_{\gamma_1}^{[\lambda_1,g_1]} \otimes \Pi_{\gamma_2}^{[\lambda_2,g_2]} = \bigoplus_{[\lambda,g],\gamma} \widetilde{N}^{\lambda,g,\gamma}_{\lambda_1,\lambda_2,g_1,g_2,\gamma_1,\gamma_2} \Pi_\gamma^{[\lambda,g]} \tag{4.59}$$

such that

$$\sum_\gamma \widetilde{N}^{\lambda,g,\gamma}_{\lambda_1,\lambda_2,g_1,g_2,\gamma_1,\gamma_2} \text{dim}(V_\gamma) = \text{dim}(V_{\gamma_1})\text{dim}(V_{\gamma_2}) N^{\lambda,g}_{\lambda_1,\lambda_2,g_1,g_2}. \tag{4.60}$$

If we restrict to the purely electric sector where $g = 0$ we still do not have an immediate agreement with the fusion rules for G. However as far as it concerns the skeleton group the restriction to trivial magnetic charge gives rise to representations of S_{el}, which is a subgroup of G. This relation will be reflected on the fusion rules as we shall see for $G = SU(2)$ in section 4.4.5.

4.4.5 FUSION RULES FOR THE SKELETON GROUP OF SU(2)

We shall compute the complete set of irreducible representations and their fusion rules for the skeleton group of $SU(2)$. From these fusion rules we shall find that the skeleton group is the maximal electric-magnetic symmetry group that can realised simultaneously in all dyonic charge sectors of the theory. Finally we compare our computations with similar results obtained by Kapustin and Saulina [85].

The skeleton group can be expressed as $W \ltimes (T \times T^*)$ modded out by a normal subgroup $D \subset W \ltimes T$ as explained in section 4.3.5. For the $SU(n)$-case W and D are computed in appendix D.1 and for $SU(2)$ they equal respectively $\mathbb{Z}_4$ and $\mathbb{Z}_2$. The latter group is precisely the centre of $SU(2)$.

The irreducible representations of S for $SU(2)$ correspond to a subset of irreducible representations of $\mathbb{Z}_4 \ltimes (T \times T^*)$ which represent D trivially. This leads to a constraint on the centraliser charges and the electric charge as given by equation (4.52).
If both the electric charge and magnetic charge vanishe the centraliser is $\mathbb{Z}_4$ which is generated by

$$x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.61}$$

The allowed centraliser representations are the two irreducible representations that send x^2 to 1. One of these representations is the trivial representation. This leads to the trivial representation of the skeleton group which we denote by $(+, [0, 0])$. The remaining non-trivial centraliser representations maps x to -1 and gives a 1-dimensional irreducible representation of the skeleton group which we shall denote by $(-, [0, 0])$.
If either the electric or the magnetic charge does not vanish the orbit under the $\mathbb{Z}_4$ action has two elements and the centraliser group is $\mathbb{Z}_2 \subset Z_4$ generated by x^2. The irreducible representations of $\mathbb{Z}_2$ that satisfies equation (4.52) is uniquely fixed by the electric charge λ labelling the equivalence class $[\lambda, g]$. It is the trivial representation if the electric charge is even and it is the non-trivial representation if the electric charge is odd. We can thus denote the resulting irreducible skeleton group representation by $[\lambda, g]$ with λ or g non-vanishing. Note that these representations are 2-dimensional.

The electric-magnetic charge sectors appearing in decomposition of a tensor product of irreducible representations of the skeleton group can be found from the fusion rules of $\mathbb{Z}[\Lambda \times \Lambda^*]$ as discussed in section 4.4.4. Ignoring the centraliser charges this gives the following fusion rules:

$$[0, 0] \otimes [0, 0] = [0, 0] \tag{4.62}$$

$$[0, 0] \otimes [\lambda, g] = [\lambda, g] \tag{4.63}$$

$$[\lambda_1, g_1] \otimes [\lambda_2, g_2] = [\lambda_1 + \lambda_2, g_1 + g_2] \oplus [\lambda_1 - \lambda_2, g_1 - g_2]. \tag{4.64}$$

To retrieve the fusion rules of the skeleton group itself one should take into account the centraliser representations. However for all charges except $[0, 0]$ the centraliser representations is uniquely determined. If we restrict to $[0, 0]$ charges we obviously obtain $\mathbb{Z}_4$ fusion rules. This leads to:

$$(s_1, [0, 0]) \otimes (s_2, [0, 0]) = (s_1 s_2, [0, 0]) \tag{4.65}$$

$$(s, [0, 0]) \otimes [\lambda, g] = [\lambda, g] \tag{4.66}$$

$$[\lambda_1, g_1] \otimes [\lambda_2, g_2] = [\lambda_1 + \lambda_2, g_1 + g_2] \oplus [\lambda_1 - \lambda_2, g_1 - g_2]. \tag{4.67}$$

If in the last line the electric-magnetic charges are parallel so that $[0, 0]$ appears at the right hand side we have to interpret this as a 2-dimensional reducible representation. Its decomposition into irreducible representations can be computed using equation (4.42):

$$\left\langle \chi_{(s,[0,0])}, \chi_{[\lambda,g]\otimes[\lambda,g]} \right\rangle = \dim(\mathbb{Z}_4 \times T \times T^*). \tag{4.68}$$

Hence we get

$$[\lambda, g] \otimes [\lambda, g] = [2\lambda, 2g] \oplus (-, [0, 0]) \oplus (+, [0, 0]). \tag{4.69}$$

An important question is if the fusion rules obtained here provide a hint about an extended electric-magnetic symmetry. The representations of such a symmetry should be uniquely labelled by the dyonic charges and do not carry additional quantum numbers. Moreover the representations with vanishing magnetic charge correspond to representations of the electric group. From this perspective the skeleton group representations $(\pm, [0, 0])$ are part of odd dimensional representations of $SU(2)$. In this way one can at least reconstruct the fusion rules of $SU(2)$ in the magnetically neutral sector. As an example we consider equation (4.69) with λ equal the the fundamental weight of $SU(2)$ and $g = 0$:

$$[\lambda, 0] \otimes [\lambda, 0] = [2\lambda, 0] \oplus (-, [0, 0]) \oplus (+, [0, 0]). \tag{4.70}$$

First we identify all representations with representations of S_{el} by “forgetting” the magnetic charge. Second we note that since $S_{el} \subset SU(2)$ as proven in section 4.3.5 the representation of the latter fall apart into irreducible representations of the former. In particular the trivial representations of $SU(2)$ can be identified with the trivial representation of S_{el} while the triplet falls apart into $[2\lambda] \oplus (-, [0])$, with 2λ equal to the highest weight of the triplet representation. Equation (4.70) is thus a simple consequence of the fact that

$$2 \otimes 2 = 3 \oplus 1. \tag{4.71}$$

One could try to push this line of thought through for $g \neq 0$. Unfortunately in this case $[2\lambda, 0]$ does not appear in equation (4.69). Adding this term by hand readily leads to problems since this forces one to add corresponding terms on left hand side. In this case one should replace $[\lambda, g]$ by $[\lambda, g] \oplus [\lambda, -g]$. This implies that $[\lambda, g]$ could never be an irreducible representation for an extended electric-magnetic symmetry containing the skeleton group since it would have to be paired with $[\lambda, -g]$ which labels an inequivalent charge sector as discussed in section 4.2.3. Consequently the skeleton group should be interpreted as the maximal electric-magnetic symmetry group that can be realised in all dyonic charge sectors. In a set of dyonic charge sectors that is closed under fusion, such as for example the magnetically neutral sectors, the electric part of skeleton group can be extended to the centraliser group in G of the magnetic charges of that particular set of sectors. One might wonder if this implies that the skeleton group does not respect gauge invariance. We shall come back to that discussion in section 4.6

Another approach to give a unified description of an electric group G and a magnetic group G^* is to consider the OPE algebra of mixed Wilson-'t Hooft operators. Such operators are labelled by the dyonic charge sectors as explained by Kapustin in [48]. Moreover the OPE's of Wilson operators are given by the fusion rules of G while the OPE's for 't Hooft operators correspond to the fusion rules of G^*. These facts were used by Kapustin and Witten [18] to prove that magnetic monopoles transform as G^*-representations. It is thus natural to ask what controls the product of mixed Wilson-'t Hooft operators. The answer must somehow unify the represntation theory of G and G^*. Consequently one might also expect it sheds some light one the fusion rules of dyons.
For a twisted $\mathcal{N} = 4$ SYM with gauge group $S0(3)$ products of Wilson-'t Hooft operators have been computed by Kapustin and Saulina [85]. In terms of dyonic charge sectors they found for example:

$$[n, 0] \cdot [0, 1] = \sum_{j=0}^{n} [n - 2j, 1]. \tag{4.72}$$

This rule can easily be understood from the fusion rules of the skeleton group for $SO(3)$ or $SU(2)$. First we note that for $G = SO(3)$ Λ can be identified with the even integers. Th magnetic weight lattice Λ^* for the $G^* = SU(2)$ is then given by $\mathbb{Z}$. The $[n, 0]$ corresponds to the magnetically neutral sector and thus to the $n + 1$-dimensional irreducible representation of $SO(3)$ or $SU(2)$. This representation falls apart into a sum irreducible representations of S_{el}. In terms of magnetically neutral representations of the skeleton group this sum of irreducibel representations is given by

$$\bigoplus_{j=0}^{n-1} [n - 2j, 0] + (s, [0, 0]). \tag{4.73}$$

Note that the centralizer label s in (4.73) depends uniquely on n. The 't Hooft operator labelled by $[0, 1]$ can be uniquely related to the irreducible representation $[0, 1]$ of the skeleton group. Simalarly for the Wilson-'t Hooft operators appearing at the right hand side of equation (4.72) there is also a unique identification with skeleton group representations. Finally we note that the decomposition of the tensor products of $[0, 1]$ with the

reducible representations (4.73) into irreducible representation of the skeleton group is given by the right hand side of equation (4.72).
A second product rule obtained in [85] which is consitent with the results [18] can be written in terms of electrially nuetral charge sectors as:

$$[0,1] \cdot [0,1] = [0,2] + [0,0] \tag{4.74}$$

This product rule cannot be understood from the fuson rules of the skelton group S of $SO(3)$. It is however consistent with the fusion rules of the dual skeleton group S^* introduced in section 4.3.5. In this particular case S^* is the skeleton group of $SU(2)$. The product rule (4.74) can be identified with the $SU(2)$ tensor product decomposition given in (4.71). It is explained above that this fusion rule is consisent with the fsuion rules of the skeleton group of $SU(2)$.
The last OPE product rule found in [85] can be represented as

$$[2n,1] \cdot [0,1] = [2n,2] + [2n,0] - [0,0] - [2n-2,0], \tag{4.75}$$

while from equation (4.67) we find for the related tensor product decompostion of skeleton group representations;

$$[2n,1] \cdot [0,1] = [2n,2] \oplus [2n,0], \tag{4.76}$$

One observes that the terms missing in this last equation correspond to the terms in (4.75) with a minus sign. However, such negative terms can occur naturally in the K-theory approach as used in [85] but can never occur in a tensor product decomposition.
We conclude that fusion rules of the skeleton group are to some extent consistent with the OPE algebra discussed by Kapustin and Saulina. The advantage of their appraoch is first that there is never need to restrict the gauge groups to certian subgroups as we effectively do with the skeleton group. Also, the OPE's of Wilson-'t hooft operators do indeed give a unified electric-magnetic algebra, whereas in the skeleton group approach one does still needs the dual skeleton group. Nonetheless, because of the occurence of negative terms the OPE algebra cannot be interpreted as a set of physical fusion rules for dyons like we can with the tensor product decompostion of the skeleton group and its dual.

4.5 S-DUALITY

To check the validity of the skeleton group we shall show that the standard S-duality action on the complex coupling of the gauge theory and the electric-magnetic charges is respected by the skeleton group. We shall first recapitulate some details of S-duality. Second, we discuss its action on the dyonic charge sectors and finally we prove that there is a well-defined S-duality action on the skeleton group representations.

4.5.1 S-DUALITY FOR SIMPLE LIE GROUPS

In for example $\mathcal{N}=4$ SYM theory S-duality leaves by definition the BPS mass invariant. The universal mass formula for BPS saturated states [3] in a theory with gauge group G can be written as:

$$M_{(\lambda,g)} = \sqrt{\frac{4\pi}{\mathrm{Im}\tau}}|v\cdot(\lambda+\tau g)|. \tag{4.77}$$

The electric charge λ takes value in the weight lattice $\Lambda(G)\subset\mathfrak{t}^*$ while g is an element in the weight lattice $\Lambda(G^*)\subset\mathfrak{t}$ of the GNO dual group. The complex coupling τ is defined as

$$\tau = \frac{\theta}{2\pi}+\frac{4\pi i}{e^2}. \tag{4.78}$$

A discussion of the action of S-duality groups on the electric-magnetic charges is discussed by Kapustin in [48], see also [86, 87]. First we choose the short coroots to have length $\sqrt{2}$, i.e.

$$\langle H_\alpha, H_\alpha\rangle = 2. \tag{4.79}$$

Now define a map ℓ acting on the CSA of G and its dual

$$\begin{aligned} \ell : H_\alpha \in \mathfrak{t} \mapsto H_\alpha^\star &= \frac{\langle H_\alpha, H_\alpha\rangle}{2}\alpha \in \mathfrak{t}^* \\ \ell^{-1} : \alpha\in\mathfrak{t}^* \mapsto \alpha^\star &= \frac{2H_\alpha}{\langle H_\alpha, H_\alpha\rangle} \in \mathfrak{t}. \end{aligned} \tag{4.80}$$

This map is implicitly used in the definition of the BPS mass formula since

$$v\cdot H_\alpha \equiv \frac{\langle H_\alpha, H_\alpha\rangle}{2} v\cdot\alpha \tag{4.81}$$

which indeed leads to the usual degeneracy in the BPS mass spectrum.
Now consider the following actions of the generators:

$$C : \tau\mapsto\tau \qquad (\lambda, g)\mapsto(-\lambda,-g) \tag{4.82}$$

$$T : \tau\mapsto\tau+1 \qquad (\lambda, g)\mapsto(\lambda-g^\star, g) \tag{4.83}$$

$$S : \tau\mapsto -\frac{1}{\tau} \qquad (\lambda, g)\mapsto(g^\star, -\lambda^\star). \tag{4.84}$$

One can easily check that the action of these generators leave the BPS mass formula (4.77) invariant. Moreover it should be clear from the action on the complex coupling that C, T and S are the familiar generators of $SL(2,\mathbb{Z})$. Unfortunately the electric-magnetic charge lattice $\Lambda(G)\times\Lambda(G^*)$ is in general not invariant under the action of $SL(2,\mathbb{Z})$. However As explained in section 4.2 it is very natural in for example a $\mathcal{N}=4$ gauge theory with smooth monopoles to take both G and G^* to be adjoint groups and thereby restrict the electric charges to the root lattice and the magnetic charges to the coroot lattice. One can show that lattice $\Lambda_r\times\Lambda_{cr}$ is invariant under some subgroup of $SL(2,\mathbb{Z})$.

A long coroot H_α is mapped to a multiple of α since the length-squared of a long coroot is an integral multiple of the length-squared for a short coroot. Consequently, the image of Λ_{cr} under ℓ is contained in the root lattice Λ_r and hence also in the weight lattice $\Lambda(G)$ of G.
We want to check if ℓ^{-1} maps the root lattice of G into the coroot lattice. Note that the long roots are mapped on the short coroots. This means that the length-squared of the image of a short roots has length-squared smaller than 2. Hence the root lattice is mapped into the coroot lattice by l^{-1} only if G is simply-laced.
One finds that the action of the generator S does not leave $\Lambda_r \times \Lambda_{cr}$ invariant in the non-simply laced case, but even then one can still consider the transformation ST^qS which acts as

$$ST^qS : (\lambda, g) \to (-\lambda, -q\lambda^\star - g). \tag{4.85}$$

For q sufficiently large $q\lambda^\star$ is always an element of the coroot lattice, hence there is a subgroup $\Gamma_0(q) \subset SL(2,\mathbb{Z})$ that generated by C, T and ST^qS that leaves $\Lambda_r \times \Lambda_{cr}$ invariant. The largest possible duality group for e.g. $SO(2n+1), Sp(2n)$ and F_4 is $\Gamma_0(2)$ while for G_2 it is $\Gamma_0(3)$.

4.5.2 S-duality on charge sectors

We have seen above that there is an action of $SL(2,\mathbb{Z})$ or at least an action of a subgroup $\Gamma_0(q)$ if we restrict the electric-magnetic charge lattice to $\Lambda_r \times \Lambda_{cr}$. The restriction of the charge lattice also defines a restriction of the dyonic charges sectors to $(\Lambda_r \times \Lambda_{cr})/\mathcal{W}$. Here we shall show that the duality transformations give a well defined action on these charge sectors.

The generators of the duality group may map (λ, g) to a different equivalence class under the action of the Weyl group and hence to a different charge sector. However the duality transformation maps all charges in one Weyl orbit to yet a single Weyl orbit as follows from the fact that the action of the generators of $SL(2,\mathbb{Z})$ commute with the diagonal action of the Weyl group [48]. This is obvious for C since $wC(\lambda, g) = w(-\lambda, -g) = (-w(\lambda), -w(g)) = Cw(\lambda, g)$. For T and $w \in \mathcal{W}$ we have: $wT(\lambda, g) = w(\lambda + g^\star, g) = (w(\lambda) + w(g^\star), w(g)) = (w(\lambda) + w(g)^\star, w(g)) = T(w(\lambda), w(g)) = Tw(\lambda, g)$. Finally for S we have $wS(\lambda, g) = w(-g^\star, \lambda^\star) = (-w(g)^\star, w(\lambda)^\star) = Sw(\lambda, g)$.

4.5.3 S-duality and skeleton group representations

Since there is a consistent action of the duality group on the dyonic charge sectors on may also try to extend this action to the set of representations of the skeleton group which are labelled by the dyonic charge sectors and by centraliser representations of the lifted Weyl group W. We shall assume that the duality action does not affect the centraliser labels. This is consistent if, one, it maps an irreducible representation to another irreducible representation and, two, if the action respects the fusion rules. Note that we are not considering all representations of the skeleton group but only those that correspond to the root and coroot lattice. Effectively we thus have modded the skeleton group out by a discrete group.

To prove the consistency of the duality group action we shall use the following ingredients. First, the action of C, T and S, and hence also the action of the duality group commutes with the action of the lifted Weyl group. This follows immediately from the fact that the duality group commutes with the Weyl group as we have shown in the previous section. Second, the centraliser subgroup in W is invariant under the action of the duality group on the electric and magnetic charge.
Since the action of $\mathcal{W}$ and thus also W on the electric-magnetic charges is linear it should be clear that charge conjugation does not change the centraliser.
The fact that T does leave the centraliser group $W_{(\lambda,g)} \subset W$ invariant is seen a follows: let $W_g \subset W$ be the centraliser of g so that for every $w \in W_g$ $w(g) = g$. The centraliser of (λ, g) consists of elements in $w \in W_g$ satisfying $w(\lambda) = \lambda$. Similarly the elements $w \in W_{(\lambda+g^\star,g)}$ satisfy $w(g) = g$ and thus $w(g^\star) = g^\star$. Finally one should have $w(\lambda + g^\star) = \lambda + g^\star$. But since $w(\lambda + g^\star) = w(\lambda) + w(g^\star)$ one finds that w must leave λ invariant. Hence $W_{(\lambda+g^\star,g)} = W_\lambda \cap W_g = W_{(\lambda,g)}$. Similarly the action of S is seen to leave the leave $W_{(\lambda,g)}$ invariant since $W_{\lambda^\star} = W_\lambda$ and $W_{-g^\star} = W_g$ so that $W_{-g^\star} \cap W_{\lambda^\star} = W_\lambda \cap W_g$.

An irreducible representation of the skeleton group is defined by an orbit in the electric-magnetic charge lattice and an irreducible representation of the centraliser in W of an element in the orbit. Since the $SL(2,\mathbb{Z})$-action commutes with the action of the lifted Weyl group a W-orbit is mapped to another W-orbit. We define the centraliser representation to be invariant under the duality transformation, this is consistent because the centraliser subgroup itself is invariant under $SL(2,\mathbb{Z})$. We thus find that an irreducible representation of the skeleton group is mapped to another irreducible representation under the duality transformations.

Finally we prove that S-duality transformations respect the fusion rules of the skeleton group. The claim is that if for irreducible representations Π_a of the skeleton group one has

$$\Pi_a \otimes \Pi_b = n_{ab}^c \Pi_c, \tag{4.86}$$

then for any element s in the duality group one should have

$$\Pi_{s(a)} \otimes \Pi_{s(b)} = n_{ab}^c \Pi_{s(c)}. \tag{4.87}$$

By inspecting equation (4.42) we can proof this equality. First we note that since s commutes with the lifted Weyl group we have for any $(\mu', h') \in [s(\lambda, g)]$ $(\mu', h') = s(\mu, h)$ for a unique $(\mu, h) \in [\lambda, g]$. In that sense the summation over the orbits $[\lambda, g]$ and $[s(\lambda, g)]$ is equivalent. Next we see that since s is an invertible linear map on the dyonic charges $s(\mu_3, g_3) = s(\mu_1, g_1) + s(\mu_2, g_2)$ if and only if $(\mu_3, g_3) = (\mu_1, g_1) + (\mu_2, g_2)$. Similarly we find $hs(\mu, g) = s(\mu, g)$ if and only if $h(\mu, g) = (\mu, g)$. Finally we note that if one defines $x_{(\mu,h)} \in W$ by $x_{(\mu,h)}(\lambda, g) = (\mu, h)$ then $x_{(\mu,h)} s(\lambda, g) = s(x_{(\mu,h)}(\lambda, g)) = s(\mu, h)$ and hence $x_{s(\mu,h)} = x_{(\mu,h)}$. Under the assumption that the S-duality action does not affect the centraliser charges we now find

$$\langle \chi_c, \chi_{a\otimes b} \rangle = \left\langle \chi_{s(c)}, \chi_{s(a)\otimes s(b)} \right\rangle . \tag{4.88}$$

This proves (4.87).

4.6 Gauge Fixing and non-abelian phases

One of the basic motivations to find fusion rules that comprises both the electric, magnetic and dyonic charge sectors, is to obtain tools to understand the phase structure of gauge theories and possible transitions between these phases. As we shall argue below dyonic phases correspond to *generalised Alice phases*. The appearance of such Alice phases may be understood from a beautiful idea of 't Hooft [29]. The philosophy is that the physically relevant parameters in a particular phase can be isolated by gauge fixing with so-called *non-propagating* gauge conditions. A well known example of such a phase-gauge relation is the Coulomb phase and the abelian gauge. In this gauge the non-abelian theory is effectively described by an abelian theory. We shall generalise this idea and argue that there is a non-propagating gauge particularly suitable to describe dyonic phases in which at long range the manifest symmetry is neither purely electric nor purely magnetic. The effective description of the non-abelian theory in this skeleton gauge is very similar to the theory in the abelian gauge except for an additional discrete symmetry generalising the charge conjugation symmetry from Alice electrodynamics [30, 31, 32].

4.6.1 The Coulomb phase and the abelian gauge

In this subsection we recall some well known facts about the relation between the Coulomb phase and the non-propagating abelian gauge for an $SU(2)$ or $SO(3)$ theory.

In a gauge theory we are confronted with an infinite (local) gauge symmetry which implies that the description of the true physical degrees of freedom can be done in different formulations, by employing different choices of gauge. What matters is that the physical observables are gauge invariant. The physically distinct phases of gauge theories therefore all respect the full local gauge invariance, in spite of the fact that their spectra may differ as well as their topological structure. As we mentioned before, to study a particular phase it can be advantageous to choose a gauge which stays close to the physics of that phase. In particular the expected low energy effective description of that phase usually suggests what this parametrisation should be. Yet, since it is only a gauge choice, it also allows one to study the other phases from this particular point of view. These gauges may offer new perspectives on some of the elusive features of gauge theories.

Important results in this direction were obtained for pure gauge theories by 't Hooft [29] by introducing the notion of a so-called non-propagating gauge, in particular the *abelian gauge*. He showed that by choosing a particular gauge fixing he could rewrite the non-abelian theory as an effective abelian theory with magnetic monopoles corresponding to the singularities in the gauge. The abelian gauge is therefore particularly suitable to study the Coulomb phase where the long-range forces are indeed abelian. Nonetheless, approximate models of this sort have been successfully implemented in certain lattice formulations to investigate the confining phase, exploiting the fact that a strongly coupled $U(1)$ with monopoles does indeed confine, through a dual Higgs effect [88].

Let us discuss the unified electric-magnetic symmetry in the abelian gauge. An abelian gauge theory with monopoles has a manifest electric gauge group $T = U(1)$ and a topological conservation law for the magnetic charge. To account for the magnetic part of the spectrum we can assume the theory to have a hidden global $T^* = U(1)$. One can also introduce a second gauge potential making T^* local, but then one has to impose a constraint to ensure that the second gauge field does not carry physical degrees of freedom, as the theory has only a single photon. This can be done and was formulated in e.g. [89] in an attempt to make a consistent quantum field theory involving both electrically charged particles and magnetic monopoles. So the conclusion is that the theory in the abelian gauge has in fact a $T \times T^*$ symmetry. However let us make an observation at this point. Even accepting the hidden T^* symmetry the spectrum of states has still a symmetry between particles with electric-magnetic charges equal up to an overall sign. This is obvious from in the original gauge theory with gauge group G as discussed in section 4.2.3. From the $T \times T^*$ perspective on the other hand the degeneracy signals the potential presence of a hidden symmetry. This symmetry is nothing but a (local) charge conjugation symmetry common from Alice phases.

4.6.2 The Alice phase and the skeleton gauge

We briefly discuss the prototype Alice phase in a theory with gauge group $G = SO(3)$ and explain how it is related to the skeleton group. Furthermore we explain how the Alice phase is related to a particular non-propagating gauge condition for $SO(3)$. Finally we generalise these concepts to a theory with an arbitrary non-abelian gauge group. Along the way we classify the singularities that can arise by gauge fixing with a non-propagating gauge condition and argue when such singularties obstruct the implementation a particluar non-propagating gauge condition in an $SU(2)$ or $SO(3)$ gauge theory.

The Alice phase can be described as a Higgs phase with a condensate in the 5-dimensional irreducible representation of $SO(3)$ [30, 31, 32]. The expectation value of the Higgs field can be identified with the traceless symmetric tensor:

$$\langle \Phi_{ij} \rangle = \eta_i \eta_j - \frac{1}{3}\delta_{ij}\eta_k\eta_k, \tag{4.89}$$

where $\eta = \eta_i \sigma_i$ and $(\sigma_i)_{i=1,\dots,3}$ the Pauli matrices. If η is nonzero the $U(1)$-group of rotations around the vector η leave by definition η invariant and hence the Higgs VEV invariant. The latter is also invariant under the reflection $\eta \mapsto -\eta$. These two parts of the residual symmetry for nonvanishing η do not commute and the full residual symmetry group H is $\mathbb{Z}_2 \ltimes U(1)$, the electric subgroup S_{el} of the skeleton group for $SO(3)$.

The topology of the model allows for monopoles just like the $U(1)$ phase because $\pi_2(G/H) = \mathbb{Z}$, but also allows a non-trivial $\mathbb{Z}_2$ flux solution because $\pi_1(G/H) = \mathbb{Z}_2$. Such a smooth flux solution has been constructed in [90], see also [91]. The ansatz for this solution is due to Schwarz [31]. In cylindrical coordinates the asymptotic Higgs field $\Phi(r, \theta, z) = \phi(\hat{r}) + \mathcal{O}(r^{-1})$ is determined in terms of η via equation (4.89) by

$$\eta_i(\theta) \sim R_{ij}(\theta/2)\eta_j(0), \tag{4.90}$$

where $\eta(0) \sim \sigma_3$ and $R_{ij}(\varphi)$ corresponds to a rotation around the x-axis over an angle φ. The $U(1)$-factor of the residual gauge group at some point on an infinitely large cylinder is generated by $\eta(\theta)$. If one goes around the flux this generator is transformed by a $\mathbb{Z}_2$-action since $\eta(2\pi) = -\eta(0)$. It is important to note that the flux solution itself is perfectly smooth at $\theta = 0$, i.e. the Higgs field, and actually also the gauge field, is a single valued function.

In general one can associate to every flux solution a specific element h in the residual gauge group defined by a Wilson loop around the flux:

$$h = Pe^{\oint A_\mu dx^\mu}. \tag{4.91}$$

The element h can in principle be any element in H, but the flux is topologically stable if it lies in a subset of H which does not contain the unit. In this particular case we thus

find that up to a $U(1)$ element this so-called Alice flux h equals the generator of $\mathbb{Z}_2$. This implies that if an electrically charged particle is moved around the Alice flux, its charge is conjugated by the $\mathbb{Z}_2$ action. The important conclusion is that it is not possible to give a single valued definition for the electric $U(1)$ charges in the presence of an Alice flux. In other words the $U(1)$ generator is not single valued and changes sign if one takes it around the $\mathbb{Z}_2$ flux as we have seen in the example above. In that sense the Alice phase is an abelian theory with a local charge conjugation symmetry. Similarly, in the presence of an Alice flux the sign of the magnetic charge is not uniquely defined. The heuristic argument for this is that the magnetic charge is defined in terms of the $U(1)$ generator and thus its sign flips if we take it around the $\mathbb{Z}_2$-flux. More precise arguments for this are based on topology [92].
Because there is only one type of $\mathbb{Z}_2$ flux it should be clear that if we take a dyon around it the electric as well as the magnetic charge changes sign and the charge conjugation acts diagonally on dyonic charges. Finally note that if under $\mathbb{Z}_2$ gauge transformations the charges change sign, also the magnetic and electric fields change sign so that (locally) the physics remains unaffected by the gauge transformations. This means that there must be two distinct one-dimensional neutral representations, one for the vacuum sector and one for the photon.

The degeneracies of the Alice phase are accurately reflected in the representation theory of the skeleton group. For non-vanishing charges the irreducible representations are 2-dimensional and the electric-magnetic charges of the states constituting such a representation are related by a $\mathbb{Z}_2$ reflection. In the neutral sector there are two irreducible representations, the trivial representation $(+, [0, 0])$ corresponding to the vacuum and the non-trivial representation $(-, [0, 0])$ related to the photon. This is important in our view, because in the abelian phase this is not the case.
Finally note that locally the physics in Alice electrodynamics is not different from ordinary electrodynamics. Nonetheless on a global level these theories are profoundly different because the Alice fluxes mediate topological interactions, see e.g. [92].

To understand the relevance of the skeleton gauge one needs some understanding of the abelian gauge. In 't Hooft's proposal a non-propagating gauge is introduced by means of some tensor X transforming in the adjoint of the gauge group. The order parameter X can either be a fundamental field of the theory or be constructed out of a composit field. In a pure Yang-Mills theory one can take for example X to be contained in the tensor product $F_{\mu\nu} \otimes F^{\mu\nu}$. Note though that in the case that G equals $SU(2)$ or $SO(3)$ the decomposition of the symmetric tensor product of the adjoint into irreducible representations does not contain the adjoint represention and one needs some other field to define X.
One can now fix a gauge by requiring X to be a diagonal matrix, i.e. by taking X in the CSA. However, to obtain the abelian gauge where the residual gauge group equals the maximal torus T one also has to fix the order of the eigenvalues. In the case of $SU(n)$

we can restrict X to be of the form diag$(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n)$. Since the eigenvalues of X are gauge invariant the abelian gauge is the strongest non-propagating gauge condition that can be implemented. If we leave out the additional constraint we obtain a non-propagating gauge condition where the eigenvalues are ordered up to Weyl transformations. In this gauge the residual symmetry of $SO(3)$ is thus the electric subgroup of the skeleton group $\mathbb{Z}_2 \ltimes U(1)$, i.e. it is a minimal non-abelian extension of the maximal torus $U(1)$.

An important aspect of the abelian gauge is that is gives rise to singularities. To see this one can start out from a configuration of X that for the sake of simplicity is smooth over $\mathbb{R}^3$, i.e. X defines a trivial adjoint bundle associated to a trivial principal G-bundle over $\mathbb{R}^3$. One may now wonder if for such a configuration there always is a smooth or at least continuous gauge transformation that rotates X into the CSA with a fixed order of eigenvalues. If this can be done one ends up with trivial T-bundle over $\mathbb{R}^3$ and in particular with a trivial T-bundle over any sphere $S^2 \subset \mathbb{R}^3$. We know already that there are configurations corresponding to a trivial G-bundle over $\mathbb{R}^3$ for which such a gauge transformation does not exist. These are related to a non-trivial T-bundles over a 2-sphere.
An example for $G = SU(2)$ (or $G = SO(3)$) directly related to the 't Hooft-Polyakov monopole [6, 7] is if X equals the "hedgehog" configuration $r_i\sigma_i h(r)$ with $h(r)$ approaching some constant value for small values of r. Note that the stabiliser of X at each point in $\mathbb{R}^3$ is a subgroup $U(1) \subset SU(2)$ generated by $\hat{r}_i\sigma_i$, except at the origin where X vanishes and the residual gauge group is restored to $SU(2)$. There is a gauge transformation that maps the hedgehog configuration to $\sigma_3 r h(r)$ which is discontinuous along the negative z-axis (including the origin). This Dirac string is just a gauge artifact as can be seen by adopting the Wu-Yang description [37]: there also exists another gauge transformation mapping $\hat{r}_i\sigma_i$ to σ_3 which is discontinuous on the positive z-axis. These two $SU(2)$ gauge transformations are related by a non-trivial $U(1)$ gauge transformation which is well-defined on $\mathbb{R}^3$ except for the z-axis. Consequently the hedgehog configuration defines a non-trivial $U(1)$ bundle one each S^2 centred around the origin. The Dirac strings are now accounted for by using a separate gauge transformation on the two hemispheres of S^2. Nonetheless the patched gauge transformation on $\mathbb{R}^3$ is still singular at the origin where the full gauge group is restored.
In general there exist smooth configurations of X which define a non-trivial winding numbers in $\pi_2(SU(2)/U(1)) \sim \pi_1(U(1))$ and can therefore not be rotated into the CSA without introducing point-like singularities where $SU(2)$ is restored. All these singularities have to be added as extra degrees of freedom to the effective theory in the abelian gauge.

It is reasonable to ask if there are not any other types of singularities which one has to add to the effective theory. We have already seen that there are string-like objects such as the Dirac string which are merely gauge artifacts. The state of any particle in the theory remains unchanged if the particle is moved around a Dirac string. Alice fluxes on the other hand are string-like objects which are truly physical. The state is transformed by a $\mathbb{Z}_2$-transformation if the particle is moved around it. Say we now start out from a smooth configuration corresponding to an Alice flux so that $X = \eta_i(r,\theta)\sigma_i\alpha(r)$ with, for large values r, η_i as defined in equation (4.90). There certainly exists a gauge transformation that rotates X into the CSA. This gauge transformation, which is essentially given by the the rotation matrix $R(\theta/2)$, is discontinuous at $\theta = 0$ and thus gives rise to a plane-like singularity bounded by the z-axis. Again this singularity is just a gauge artifact that can be circumvented by using a two-patched gauge transformation. One now obtains a non-trivial bundle on each cylinder centred around the z-axis which is essentially a Möbius strip. But even in this description the singularity at the z-axis itself remains and hence the Alice flux has to be added to to effective theory in the non-propagating gauge just as is the case with monopoles. The appearance of such a string-like object was in some sense already foreseen by ’t Hooft, see section 3 of [29].

It is nice to note that at the string singularity the full $SU(2)$ group is not restored. Instead the stabiliser of X equals the $U(1)$ group generated by σ_1 [91]. What is crucial, however, is that X is not single valued as we already noted earlier. This means that though in an Alice flux configuration X can be rotated into the CSA, its eigenvalues cannot be ordered since they are permuted as one goes around the flux. Both orderings appear simultaneously. Consequently the abelian gauge cannot be implemented in the the background of an Alice flux, the strongest gauge condition that can be used is the skeleton gauge in which the ordering condition is left out and the gauge is fixed only by taking X to be diagonal. If one insists on using the abelian gauge this means one has to disregard Alice flux configurations thereby ignoring a relevant sector of the theory in which its non-abelian nature is manifest.
Another way to implement the skeleton gauge is not to use an order parameter in the adjoint representation but instead an order parameter in the 5-dimensional irreducible representation. As follows from the discussion earlier in this section such an order parameter is single valued in an Alice flux background. Moreover this order parameter can also be used in a pure $SU(2)$ or $SO(3)$ Yang-Mills theory because, as opposed to the adjoint representation, the 5-dimensional representation appears in the decomposition of the symmetric part of $3 \otimes 3$. Also note that for non-zero values of X the residual gauge symmetry equals the electric part of the skeleton group which is $\mathbb{Z}_2 \ltimes U(1)$ in the $SO(3)$ case. General smooth configurations of such an X in the 5-dimensional representations which correspond to a non-trivial equivalence class in $\pi_1(SO(3)/(\mathbb{Z}_2 \times U(1)) = \pi_0(\mathbb{Z}_2)$ give rise to string-like singularities which have to be added to the effective theory in the skeleton gauge. Also point-like singularities appearing in the skeleton gauge have to be taken into account. The smooth configurations from which these singularities arise define non-trivial elements in $\pi_2(S0(3)/(\mathbb{Z}_2 \times U(1)) = \pi_1(\mathbb{Z}_2) \times \pi_1(U(1))$. Since $\pi_1((Z_2))$ is trivial such point-like singularties are directly related to the monopole singularities in the

abelian gauge with the difference that the sign of the monopole charge is only defined up to a sign in an Alice flux background.

Besides point-like and string-like singularities there might also be plane-like singularities in the skeleton gauge or in the abelian gauge. We have already seen such singularities in the skeleton gauge which correspond to a discontinuity of the $U(1)$-generator in the flux background. Such singularties can only be truly physical if they arise from a smooth configuration defining a non-trivial element in $\pi_0(G/H)$. Since G is connected in the case we consider here, this homotopy group vanishes. That tells us that all plane-like singularities must be gauge artifacts.

Just as the abelian gauge cannot always implemented for Alice flux configurations there may be configurations which obstruct the implementation of the skeleton gauge. Such configurations do indeed exist and correspond to topologically stable flux solutions in theories where the gauge group is broken a discrete (non-abelian) subgroup of $SO(3)$ which is not a subgroup of $\mathbb{Z}_2 \ltimes U(1)$. If one goes around such a flux, $\eta = \eta_i \sigma_i$ is transformed to Ad$(h)(\eta)$ with $h \notin \mathbb{Z}_2 \ltimes U(1)$ and $\eta \sim \sigma_3$ is mapped out of the CSA. Note however that a suitabale gauge fixing parameter that is single valued in the presence of such a flux, transforms in some higher dimensional representations and can in general thus be not constructed as a composite field in a pure Yang-Mills theory. Moreover, even if a suitable parameter X can be constructed or if X corresponds to a fundamental field, monopoles will be confined in such a flux sector because the flux configuration corresponds to an electric condensate that manifestly breaks $U(1) \subset SO(3)$. We thus have to conclude that even though the skeleton gauge does not probe all sectors of the theory and thereby does not show its complete non-abelian symmetry, it is sufficient to describe all dyonic charge sectors.

The question is what theory is effectively described by the $SO(3)$ Yang-Mills theory in the skeleton gauge. The skeleton gauge is related to the skeleton phase as the abelian gauge is related to the Coulomb phase. Hence the obvious candidate is Alice electrodynamics. So it is an abelian theory together with a local conjugation symmetry. But just as in the abelian gauge the theory contains extra degrees of freedom corresponding to magnetic monopoles. In the skeleton gauge the effective theory also involves an Alice flux. We thus see that even though the residual gauge group is non-abelian the effective theory should still be "manageable" since the non-abelian factor is only manifest via topological interactions mediated by the Alice flux.

The Alice phase of $SO(3)$ as well as the skeleton gauge are easily generalised to other gauge groups. A generalised Alice phase is per definition a gauge theory with gauge group G broken to its electric skeleton group S_{el}. It has an abelian subgroup corresponding to the maximal torus T but in addition features non-trivial (non-abelian) topological

interactions. These topological interactions are induced by discrete Alice fluxes which correspond to the disconnected components of the skeleton group. Note that two elements in the lift of the Weyl group $W \subset S_{el}$ are connected if they differ by an element in T. Hence the Alice fluxes can be labelled by elements in the Weyl group $\mathcal{W} = W/D$ of G, where $D = W \cap T$.
The appropriate Higgs field which generalises the symmetric traceless Higgs used in the $SO(3)$ case is described as follows. If λ is the highest weight of the adjoint representation of G then the Higgs should transform in the irreducible representation with highest weight 2λ. The connection with the $SO(3)$ case is seen from the fact that the 2λ representation is part of the symmetric tensor product Sym$(\lambda \otimes \lambda)$. Moreover to prove that this Higgs does indeed break G to S_{el} one can use the fact that each $SU(2)$ or $SO(3)$ subgroup corresponding to a simple root is broken down to $(\mathbb{Z}_4 \ltimes U(1))/\mathbb{Z}_2$ or $\mathbb{Z}_2 \ltimes U(1)$.
The set of monopoles in the generalised Alice phase is identical to the set of monopoles in the Coulomb phase where the residual symmetry equals T. The difference with the Coulomb case lies in the fact that because of the presence of Alice fluxes, monopoles with charges related by Weyl transformations should be identified. We thus find that the unified electric-magnetic symmetry in a generalised Alice phase corresponds to the skeleton group of G.

To introduce the generalised skeleton gauge one starts just as in the $SO(3)$-case with an adjoint tensor X and imposes the non-propagating gauge condition that this tensor should be diagonal. As opposed to 't Hooft's proposal we do not give any additional ordering conditions. This implies that the eigenvalues of X are ordered up to transformations given by the Weyl group $\mathcal{W}$. Consequently the residual gauge symmetry in the skeleton gauge is given by the maximal torus and in addition some discrete group which does not commute with T. The elements of this discrete group are elements in G acting on the eigenvalues of X as the Weyl group, i.e. they are elements in the lift W of $\mathcal{W}$. The total residual gauge group coincides with $(W \ltimes T)/D$, the electric subgroup of the skeleton group. A more suitable gauge fixing parameter to define the skeleton gauge would be to take X in the same representation as the Higgs field used to introduce the generalised Alice phase, since this order parameter is invariant under the action of the Weyl group.

We now claim that the effective theory for the Yang-Mills theory with gauge group G in the skeleton gauge is a gauge theory generalising the Alice theory described above. This theory is up to (lifted) Weyl transformations an abelian theory with additional monopoles. The non-abelian character of the theory is locally invisible and can only be observed in topological interactions mediated by the expected Alice strings.
From a fundamental point of view, where we want to uncover the dual electric magnetic structure of non-abelian gauge theories it now becomes clear that compared to the abelian gauge the skeleton gauge offers a richer perspective on all dyonic charge sectors of the theory. It explicitly reflects the non-abelian nature of the original theory by organising the

spectrum of magnetically and electrically charged fields in complete Weyl orbits which as we know are indeed degenerate. Also the neutral sectors acquire distinctive quantum numbers corresponding to irreducible representations of the Weyl group $\mathcal{W}$. Since it is merely just a gauge choice the skeleton gauge may in principle also be used to study other phases than generalised Alice phases.

4.6.3 PHASE TRANSITIONS: CONDENSATES AND CONFINEMENT

In this subsection we want to determine phases of an $SO(3)$ gauge theory related to an Alice phase and study the effect of condensates in sectors labelled by the representations of the skeleton group. Here we encounter an interesting interplay between group breaking and topological features, leading to an understanding why certain parts of the spectrum are “swallowed” by the vacuum, while others become confined. In the remainder of this section we systematically analyse a number of conceivable condensates and describe the phases associated with them.

Starting from the generalised Alice phase, let us consider the case where the neutral vector particle condenses, i.e. a condensate in the sector $(-, [0, 0])$. Such a condensate breaks S to $T \otimes T^*$. This phase is indeed different from the original Alice phase because the Alice string is confined. This means that possible closed loops of Alice string, which had an energy proportional to length in the skeleton phase, become pancakes with energy proportional to the minimal area spanned by the loop. The topological argument is simple: in the new phase topological domain walls form, these are labelled by $\pi_0(G/H) = \pi_0(S/(T \times T^*)) = \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$, and the Alice loops will become the boundaries of these walls. Since the Alice fluxes disappear from the bulk opposite electric and magnetic charges are no longer equivalent and therefore the skeleton representations split into $T \times T^*$ representations. This is the Coulomb phase, and as such indeed nothing but the abelian gauge description of the $SU(2)$ theory given by ’t Hooft as discussed above.

Let us now assume that in addition a charged vector boson in the representation $(2, 0)$ condenses. Note that this consenstate breaks $T \subset S$ completely and hence we end up in a Higgs phase. The residual symmetry group is given by T^* and topology is changed since now one has a spectrum of magnetic fluxes corresponding to

$$\pi_1((T \times T^*)/T^*) = \pi_1(T) = \mathbb{Z}. \tag{4.92}$$

This implies that the theory is in a phase where magnetic fluxes are forced into magnetic flux-tubes. These flux tubes match the allowed magnetic charges in the theory and one should expect all monopoles to become confined. More precisely: the minimal confined flux equals one fundamental flux quantum corresponding to the fundamental weight of the

magnetic dual group $SU(2)$. The derivation of this fact is closely related to the derivation of the Dirac quantisation condition for the allowed magnetic charges as discussed in section 2.3.1. If we move a particle with electric charge $\lambda \in \mathbb{Z}$ around a flux tube with flux $e^{ig\pi} \in U(1)$ the state of the system picks up a phase factor $e^{i\lambda g\pi}$. Hence if we move the charged vector boson around a fundamental flux tube with $g = 1$, the state of the system is single valued and one can consistently think of the vector boson a absorbed by the vacuum. This observation about single-valuedness holds of course for any particle whose electric-magnetic charge is an integer multiple of $(2, 0)$ and hence if a charged vector boson condenses the complete purely electric sector actually condenses. A dyon with charge $(\lambda, g_))$, where $g \neq 0$, becomes confined. This follows from the fact that such a dyon originates from the tensor product representation $(\lambda, 0) \otimes (0, g)$.
Note that a monopole with unit charge will have a single flux tube attached which reaches to infinity, while a monopole with charge 2, corresponding to the root of $SU(2)$, can have two fluxtubes. These observations are the three-dimensional analogue of what is encountered in two-dimensional situations studied in for example [93].
It is interesting to see what would happen on the boundary. The unit flux in the bulk defines the unit flux on the boundary. In the boundary theory the monopole is an instanton in the sense that tunnelling of a unit flux is allowed via a monopole-antimonopole pair in the bulk. Consequently on the boundary there is no non-trivial magnetic flux sector. Since in the bulk theory all electric charges are condensed, all charge sectors in the boundary theory are identified with the vacuum. The boundary theory is thus trivial.
We shall see that the boundary theory becomes non-trivial if we consider a bulk condensate in the $(4, 0)$-representation which breaks $T \times T^*$ to $\mathbb{Z}_2 \times T^*$. The minimal confined flux is now half the unit flux; if we move a $(4, 0)$ state around this minimal flux the state of the system does not pick up a non-trivial phase factor and $(4, 0)$ can consistently be identified with the vacuum representation $(0, 0)$. Note that this actually holds for any $(4n, 0)$-representation as consistent with the fact that gauge group is broken down to $\mathbb{Z}_2 \times T^*$. We again see that all monopoles are confined but the unit monopole has two flux tubes attached instead of only one, which implies that the minimal confined flux tube cannot break up into a series of monopole-anti monopole pairs. This gives rise to a non-trivial flux sector with half a unit flux. Together with the tunneling of unit fluxes via monopole-antimonopoles pairs in the bulk this gives rise to a magnetic $\mathbb{Z}_2$ factor of the boundary theory.
To understand the electric content of the boundary theory we first make some observations about the bulk theory. The condensation of electrically charged particles means that at long range their charges are screened. Since all such particles interact via the same Coulomb field we should actually say that this field and thereby all Coulomb interactions are screened at large disctances. Nonetheless, as proven in the two-dimensional case [94], one should expect all topological interactions, mediated by the magnetic fluxes, to survive at longe range. If we move a particle with electric charge $2n$ around the minimally confined flux tube, the state of the system pick up the phase factor $e^{\frac{i2\pi n}{2}}$ which is non-trvial

for $n = 1$ modulo 2. Particles that can pick up a non-trivial phase factor are not condensed and give rise to one non-trivial charge sector in the boundary theory. The complete picture is that on the boundary a $\mathbb{Z}_2$ discrete gauge theory [34]. This is consistent with the arguments which one may separately apply to the boundary theory [93].

We should make the following remark: there are phases that are difficult to describe in this language, or better in this gauge. For example the non-abelian phases, corresponding to discrete gauge theories. Certain $\mathbb{Z}_n$ and D_n models are accessible because their group can be embedded in S_{el}. The other electric discrete non-abelian phases can be and have been studied starting from the electric theory [34].

One could also break the T^* by a monopole condensate in say the $(0, 2)$ representation of $T \times T^* \subset S$. Now electric flux tubes develop which are quantised that match the possible electric charges in the theory. In this phase one should thus expect that electric charges are confined.
Here applies a fortiori what we said about the non-abelian discrete phases. We have not proven the existence of a non-abelian purely magnetic symmetry, though undoing the abelian gauge to the magnetic side one could imagine to obtain such a phase. Our findings in previous sections about the skeleton group and its dual are certainly consistent with this hypothesis.
We also want to see what happens with dyonic condensates. Let us define an exterior product notation between two representations: $[a, b] \wedge [c, d] = ad - bc$. If we identity the weight lattice of $SO(3)$ with the even integers we can write the generalised Dirac condition for dyons simply as the condition that the wedge product equals $2n$ for some $n \in \mathbb{Z}$. The condition for confinement given a condensate of $[a, b]$ is now that a representation $[c, d]$ will be confined if $|[a, b] \wedge [c, d]| \geq 2$. This is so because exactly those combined electric-magnetic fluxes allow a single valued vacuum state. Conversely, the only representations that are *not* confined are those for which the exterior product with the condensate vanishes, which implies the condition $ad - bc = 0$ or $a/b = c/d$, in other words the electric-magnetic vectors have to be parallel. The generalisation to higher dimensions should be similar but now we have the inner products between vectors on Λ and Λ^* in the exterior product: $[\vec{a}, \vec{b}] \wedge [\vec{c}, \vec{d}] = \vec{a} \cdot \vec{d} - \vec{b} \cdot \vec{c}$. The condition for confinement that the norm of the exterior product must be larger or equal than 2 remains roughly the same. Note however that the condition to not be confined allows for many more solutions in this situation.

The attentive reader will no doubt have noticed that we have "overlooked" one possible phase that should be part of our analysis. The question is what happens if in our Alice phase the Alice strings themselves condense? This possibility has been considered before [95], but the physical implications of such a condensate were not. The crucial property in the unbroken phase is that electric and magnetic charges can de-localise into so-called

Cheshire charges, which are rings of Alice flux carrying a non-localised charge, meaning that any closed surface containing the ring may contain an electric or magnetic charge, but it cannot be localised any further. This makes clear what the situation is like when the Alice strings condense: both the notions of electric and magnetic charge lose their physical meaning. Another way of saying this would be to say that both types of charge can spread (non-localisable) and neutralise any source; this means that both electric and magnetic charges will be completely screened. The particles survive as neutral particles. Another way to look at this is from the skeleton group breaking point of view [93] where oppositely charged representations would be identified in the effective low energy theory, and the photon would acquire some screening mass linked to the VEV of the Alice loop operator.

We finally note that it is very interesting to investigate phases that emerge from a condensate that partially breaks the (lifted) Weyl group symmetry in the case of higher rank groups.

APPENDIX A

THE ALGEBRA UNDERLYING THE MURRAY CONE

As announced in section 3.1.4 we shall construct an algebraic object whose set of irreducible representations corresponds to the fundamental Murray cone. We do this in such a way that the fusion rules respect the fusion rules of the residual dual group H^*. We shall start out from what is roughly speaking the group algebra of H^*. Next we introduce its dual $F(H^*)$. By dualizing again we find an object $F^*(H^*)$ which again should be thought of as the group algebra of H^*. The difference, however, is that in this new form the group algebra can explicitly be truncated to $F^*_+(H^*)$ in such a way that the irreducible representations are automatically restricted to the fundamental Murray cone. The nice feature of our construction is that it is very general. Starting out from any Lie group and any subset of irreducible representations closed under fusion we can construct a bi-algebra which has a full set of irreducible representations corresponding to the subset one started out with and whose fusion rules match those of the group one started out with. At the end we briefly discuss the group-like object H^*_+ which has the same irreducible representations and the same fusion rules as $F^*_+(H^*)$. For most common consistent truncations of the weight lattice of H^* one knows that H^*_+ is obtained from H^* by modding out a finite group. If one restricts the weight lattice to the Murray cone, however, H^*_+ is not a group any more.

As a group H^* has a natural product and coproduct:

$$h_1 \times h_2 = h_1 h_2 \tag{A.1}$$

$$\Delta(h) = h \otimes h. \tag{A.2}$$

In addition there is a natural unit 1, co-unit ϵ and antipode S by

$$1 = e \in H^* \tag{A.3}$$

$$\epsilon : h \mapsto 1 \in \mathbb{C} \tag{A.4}$$

$$S : h \mapsto h^{-1}. \tag{A.5}$$

For a finite group one can immediately define the linear extensions of these maps on the group algebra of H^*. For a continuous groups there are several ways to define a vector space with this Hopf algebra structure. We shall circumvent this discussion by considering another algebra which is manifestly seen to be a vector space. This is the Hopf algebra corresponding to the matrix entries of the irreducible representations of H^*. Let π^λ be such a representation. For the matrix entries we have:

$$\pi^\lambda_{mm'} : h \in H^* \to (\pi^\lambda(h))_{mm'} \in \mathbb{C}. \tag{A.6}$$

The set of finite linear combinations of such maps is obviously a vector space. The resulting set turns out to be a Hopf algebra and inherits a natural product, coproduct, co-unit and antipode from H^*.
The product in $F(H^*)$ is directly related to the product of representations and can thus be expressed in terms of Clebsch-Gordan coefficients, see for example chapter 3 of [96] for the SU(2) case. The coproduct is much simpler because it merely reflects the fact that the product of H^* is respected by the representations. To see this note that

$$\begin{gathered}\pi^{\lambda_1}_{m_1 m_1'} \times \pi^{\lambda_2}_{m_2 m_2'}(h) = \pi^{\lambda_1}_{m_1 m_1'} \otimes \pi^{\lambda_2}_{m_2 m_2'}(\Delta(h)) = \\ \pi^{\lambda_1}_{m_1 m_1'} \otimes \pi^{\lambda_2}_{m_2 m_2'}(h \otimes h) = \pi^{\lambda_1}_{m_1 m_1'}(h) \pi^{\lambda_2}_{m_2 m_2'}(h) = \\ \sum_\lambda C^{\lambda_1 \lambda_2 \lambda}_{m_1, m_2, m_1+m_2} C^{\lambda_1 \lambda_2 \lambda}_{m_1', m_2', m_1'+m_2'} \pi^\lambda_{m_1+m_2, m_1'+m_2'}(h)\end{gathered} \tag{A.7}$$

and

$$\begin{gathered}\Delta(\pi^\lambda_{mm'})(h_1 \otimes h_2) = \pi^\lambda_{mm'}(h_1 \times h_2) = \\ \pi^\lambda_{mm'}(h_1 h_2) = \sum_s \pi^\lambda_{ms}(h_1) \pi^\lambda_{sm'}(h_2) = \sum_s \pi^\lambda_{ms} \otimes \pi^\lambda_{sm'}(h_1 \otimes h_2).\end{gathered} \tag{A.8}$$

The product and coproduct on $F(H^*)$ are thus completely defined by:

$$\pi^{\lambda_1}_{m_1 m_1'} \times \pi^{\lambda_2}_{m_2 m_2'} = \sum_\lambda C^{\lambda_1 \lambda_2 \lambda}_{m_1, m_2, m_1+m_2} C^{\lambda_1 \lambda_2 \lambda}_{m_1', m_2', m_1'+m_2'} \pi^\lambda_{m_1+m_2, m_1'+m_2'} \tag{A.9}$$

$$\Delta(\pi^\lambda_{mm'}) = \sum_s \pi^\lambda_{ms} \otimes \pi^\lambda_{sm'}. \tag{A.10}$$

To find the unit $1 \in F(H^*)$ and the co-unit of $F(H^*)$ we note that these are defined in terms of their dual counterparts by:

$$1(h) = \epsilon(h) = 1 \in \mathbb{C} \tag{A.11}$$

$$\epsilon(\pi^{\lambda}_{mm'}) = \pi^{\lambda}_{mm'}(e) = \delta_{mm'} \in \mathbb{C}. \tag{A.12}$$

From the first equation it follows that the unit in $F(H^*)$ is given by the matrix entry of the trivial irreducible representation of H^* while the co-unit of $F(H^*)$ is related to the entries of the unit matrix in the λ-representation of H^*.
The antipode of $F(H^*)$ is defined by $S(\pi^{\lambda}_{mm'})(h) = \pi^{\lambda}_{mm'}(h^{-1})$. In the end, however, we will only be interested in the bi-algebra structure. To avoid unnecessary complication we will ignore the antipode.

To retrieve the group algebra of H^* we shall again take the dual $F^*(H^*)$ of $F(H^*)$. This space of linear functionals is generated by the basis elements $f^{ll'}_{\mu}$. These are defined in the standard way by:

$$f^{ll'}_{\mu} : \pi^{\lambda}_{mm'} \in F(H^*) \mapsto f^{ll'}_{\mu}(\pi^{\lambda}_{mm'}) = \delta_{\mu\lambda}\delta_{lm}\delta_{l'm'} \in \mathbb{C}. \tag{A.13}$$

The product and coproduct of $F^*(H^*)$ can be defined in terms of their counterparts in $F(H^*)$.

$$\begin{aligned}
&f^{l_1 l_1'}_{\mu_1} \times f^{l_2 l_2'}_{\mu_2}(\pi^{\lambda}_{mm'}) = f^{l_1 l_1'}_{\mu_1} \otimes f^{l_2 l_2'}_{\mu_2}(\Delta(\pi^{\lambda}_{mm'}))\\
&= f^{l_1 l_1'}_{\mu_1} \otimes f^{l_2 l_2'}_{\mu_2}\left(\sum_s \pi^{\lambda}_{ms} \otimes \pi^{\lambda}_{sm'}\right) = \sum_s f^{l_1 l_1'}_{\mu_1}(\pi^{\lambda}_{ms}) f^{l_2 l_2'}_{\mu_2}(\pi^{\lambda}_{sm'})\\
&= \sum_s \delta_{\mu_1\lambda}\delta_{l_1 m}\delta_{l_1' s}\delta_{\mu_2\lambda}\delta_{l_2 s}\delta_{l_2' m'} = \delta_{\mu_1\mu_2}\delta_{l_1' l_2}\delta_{\mu_2\lambda}\delta_{l_1 m}\delta_{l_2' m'}\\
&= \delta_{\mu_1\mu_2}\delta_{l_1' l_2} f^{l_1 l_2'}_{\mu_2}(\pi^{\lambda}_{mm'})
\end{aligned} \tag{A.14}$$

$$\begin{aligned}
&\Delta(f^{ll'}_{\mu})(\pi^{\lambda_1}_{m_1 m_1'} \otimes \pi^{\lambda_2}_{m_2 m_2'}) = f^{ll'}_{\mu}(\pi^{\lambda_1}_{m_1 m_1'} \times \pi^{\lambda_2}_{m_2 m_2'})\\
&= \sum_{\lambda} C^{\lambda_1\lambda_2\lambda}_{m_1,m_2,m_1+m_2} C^{\lambda_1\lambda_2\lambda}_{m_1',m_2',m_1'+m_2'} f^{ll'}_{\mu}(\pi^{\lambda}_{m_1+m_2,m_1'+m_2'})\\
&= \sum_{\lambda} C^{\lambda_1\lambda_2\lambda}_{m_1,m_2,m_1+m_2} C^{\lambda_1\lambda_2\lambda}_{m_1',m_2',m_1'+m_2'} \delta_{\mu\lambda}\delta_{l,m_1+m_2}\delta_{l',m_1'+m_2'}\\
&= C^{\lambda_1\lambda_2\mu}_{m_1,m_2,m_1+m_2} C^{\lambda_1\lambda_2\mu}_{m_1',m_2',m_1'+m_2'} \delta_{l,m_1+m_2}\delta_{l',m_1'+m_2'}\\
&= \sum C^{\mu_1\mu_2\mu}_{l_1,l_2,l} C^{\mu_1\mu_2\mu}_{l_1',l_2',l'} \delta_{l,l_1+l_2}\delta_{l',l_1'+l_2'}\delta_{\mu_1\lambda_1}\delta_{l_1 m_1}\delta_{l_1' m_1'}\delta_{\mu_2\lambda_2}\delta_{l_2 m_2}\delta_{l_2' m_2'}\\
&= \sum C^{\mu_1\mu_2\mu}_{l_1,l_2,l} C^{\mu_1\mu_2\mu}_{l_1',l_2',l'} \delta_{l,l_1+l_2}\delta_{l',l_1'+l_2'} f^{l_1 l_1'}_{\mu_1}(\pi^{\lambda_1}_{m_1 m_1'}) f^{l_2 l_2'}_{\mu_2}(\pi^{\lambda_2}_{m_1 m_1'})\\
&= \sum C^{\mu_1\mu_2\mu}_{l_1,l_2,l} C^{\mu_1\mu_2\mu}_{l_1',l_2',l'} \delta_{l,l_1+l_2}\delta_{l',l_1'+l_2'} f^{l_1 l_1'}_{\mu_1} \otimes f^{l_2 l_2'}_{\mu_2}(\pi^{\lambda_1}_{m_1 m_1'} \otimes \pi^{\lambda_2}_{m_2 m_2'})
\end{aligned} \tag{A.15}$$

The product and coproduct on $F^*(H^*)$ are thus completely defined by:

$$f_{\mu_1}^{l_1 l_1'} \times f_{\mu_2}^{l_2 l_2'} = \delta_{\mu_1 \mu_2} \delta_{l_1' l_2} f_{\mu_2}^{l_1 l_2'} \tag{A.16}$$

$$\Delta(f_{\mu}^{ll'}) = \sum C_{l_1, l_2, l}^{\mu_1 \mu_2 \mu} C_{l_1', l_2', l'}^{\mu_1 \mu_2 \mu} \delta_{l, l_1 + l_2} \delta_{l', l_1' + l_2'} f_{\mu_1}^{l_1 l_1'} \otimes f_{\mu_2}^{l_2 l_2'}, \tag{A.17}$$

where the sum in the last line is over $l_1, l_2, l_1', l_2', \mu_1$ and μ_2. Note that this sum has an infinite number of non-vanishing terms corresponding to the pairs of irreducible representations (μ_1, μ_2) whose tensor product contains the irreducible representation of H^* labelled by μ.
One can easily check that the unit and co-unit of $F^*(H^*)$ are given by:

$$1 = \sum_{\mu, l} f_{\mu}^{ll} \tag{A.18}$$

$$\epsilon(f_{\mu}^{ll'}) = \delta_{\mu 0} \delta_{l 0} \delta_{l' 0}. \tag{A.19}$$

Just as the coproduct, the unit is not properly defined because it is a sums over an infinite number of basis elements with non-vanishing coefficients. This is just a formal problem because the finite dimensional representations of $F^*(H^*)$ will only pick out a finite number of elements as we shall see below.

We can now truncate $F^*(H^*)$ to $F_+^*(H^*)$ by projecting out all functionals $f_{\mu}^{mm'}$ that do not satisfy the Murray condition for $G \to H$. This means that we will project out all functionals with μ not in the Murray cone Λ_+. The Murray condition can thus be implemented by using the following linear projection operator:

$$P : f_{\mu}^{mm'} \mapsto P(f_{\mu}^{mm'}) = \begin{cases} 0 & \text{if } \mu \notin \Lambda_+ \\ f_{\mu}^{mm'} & \text{if } \mu \in \Lambda_+ \end{cases} \tag{A.20}$$

The product and coproduct of this truncated bi-algebra are given by:

$$f_{\mu_1}^{l_1 l_1'} \times f_{\mu_2}^{l_2 l_2'} = \delta_{\mu_1 \mu_2} \delta_{l_1' l_2} f_{\mu_2}^{l_1 l_2'} \tag{A.21}$$

$$\Delta(f_{\mu}^{ll'}) = \sum C_{l_1, l_2, l}^{\mu_1 \mu_2 \mu} C_{l_1', l_2', l'}^{\mu_1 \mu_2 \mu} \delta_{l, l_1 + l_2} \delta_{l', l_1' + l_2'} P(f_{\mu_1}^{l_1 l_1'}) \otimes P(f_{\mu_2}^{l_2 l_2'}). \tag{A.22}$$

Similarly, we have for the unit and co-unit:

$$1 = \sum_{\mu, l} P(f_{\mu}^{ll}) \tag{A.23}$$

$$\epsilon(f_{\mu}^{ll'}) = \delta_{\mu 0} \delta_{l 0} \delta_{l' 0}. \tag{A.24}$$

We shall now turn to the representations of $F^*(H^*)$ and $F_+^*(H)$. First we will introduce the set $\{\pi^{\lambda}\}$ of representations of $F^*(H^*)$, where $\{\lambda\}$ is the set of irreducible representations of H^*. We define these representations of $F^*(H^*)$ by:

$$\pi^{\lambda} : f_{\mu}^{ll'} \mapsto \pi^{\lambda}(f_{\mu}^{ll'}), \tag{A.25}$$

where the matrix entries are given by

$$\left(\pi^\lambda(f_\mu^{ll'})\right)_{mm'} = f_\mu^{ll'}(\pi^\lambda_{mm'}). \tag{A.26}$$

This definition ensures that the representations $\{\pi^\lambda\}$ respect the product of $F^*(H^*)$. The representations π^λ defined here can thus be identified with the irreducible representations of H^*. Below we shall prove that π^λ itself is actually an irreducible representation of $F^*(H^*)$ and moreover we will find that these representations constitute the full set of irreducible representation of $F^*(H^*)$.

Next we want to consider if the representations π^λ of $F^*(H^*)$ are also representations of the truncated algebra $F^*_+(H^*)$. For $F^*(H^*)$ the representations above all satisfy

$$\pi^\lambda(1) = \pi^\lambda\left(\sum f_\mu^{ll}\right) = \mathbb{I}. \tag{A.27}$$

However, in $F^*_+(H^*)$ we find for the representations labelled by λ not satisfying the Murray condition:

$$\pi^\lambda(1) = \pi^\lambda\left(\sum P(f_\mu^{ll})\right) = 0. \tag{A.28}$$

Since such π^λ does not respect the identity this is not a representation of $F^*_+(H^*)$. Hence for $F^*_+(H^*)$ we must restrict to the truncated set of representations $\{\pi^\lambda\}$ satisfying the Murray condition. This last set of representations is obviously in one-to-one relation with the magnetic charges in the fundamental Murray cone for $G \to H$. Below we shall prove that this is the complete set of irreducible representations of $F^*_+(H^*)$.

We will construct the irreducible representations of $F^*(H^*)$ and $F^*_+(H^*)$ out of the representations of a set of subalgebras. Let F^* denote either $F^*(H^*)$ or $F^*_+(H^*)$. The subalgebras denoted by $F^*_\lambda \subset F^*$ are generated by $\{f_\lambda^{ll'}\}$ with fixed dominant integral weight λ. In the case of $F^*_+(H^*)$ we of course restrict λ to be a dominant integral weight in Λ_+. Note that $\cup_\lambda F^*_\lambda = F^*$. It follows from the product rule (A.16) or (A.21) that F^*_λ is indeed closed under multiplication. The identity 1_λ in F^*_λ is expressed as:

$$1_\lambda = \sum_l f_\lambda^{ll}. \tag{A.29}$$

These elements $1_\lambda \in F^*$ satisfy:

$$f \times 1_\lambda = 1_\lambda \times f \quad \forall f \in F^* \tag{A.30}$$

$$\sum_\lambda 1_\lambda = 1_{F^*} \tag{A.31}$$

$$1_\lambda \times 1_{\lambda'} = \delta_{\lambda\lambda'} 1_\lambda. \tag{A.32}$$

We can use these properties to characterize the irreducible representations of F^*. Let V be any irreducible representation of F^*. It is easy to see that for any λ the image V_λ

of V under the action of 1_λ is itself a representation of F^*. This follows from the fact that any $f \in F^*$ commutes with 1_λ as expressed by equation (A.30). V thus contains invariant subspaces $\{V_\lambda\}$. For irreducible representations all invariant subspaces must be trivial, i.e. equal either $\{0\}$ or V. Since any representation of F^* respects the identity 1_{F^*} we find from (A.31) that for at least one λ we must have $V_\lambda \neq \{0\}$, hence $V_\lambda = V$. Note that λ is unique since $V_{\lambda'} = \{0\}$ for $\lambda' \neq \lambda$ as follows from (A.32). Consequently any irreducible representation of F^* is labelled by a dominant integral weight λ. It now follows from the product rule of F^* that any $f^{ll'}_{\lambda'} \in F^*_{\lambda'}$ with $\lambda' \neq \lambda$ acts trivially on V_λ. An irreducible representation of F^* thus corresponds to an irreducible representation of F^*_λ. Fortunately the irreducible representations of F^*_λ are easily found.

Note that the labels l and l' of F^*_λ take integer values in $\{1, \ldots, n\}$ where n is the dimension of the irreducible representation π^λ of H^*. As it turns out F^*_λ is an $n \times n$ matrix algebra and it is a well known fact that such an algebra has a unique irreducible representation of dimension n. For completeness we shall prove this now.
F^*_λ has a commutative subalgebra $F^{*\,\mathrm{diag}}_\lambda$ generated by the elements f^{ll}_λ. Let us construct the irreducible representations of $F^{*\,\mathrm{diag}}_\lambda$. Since the algebra is commutative its irreducible representations are 1-dimensional. Let π be such a representation. From $\pi(f^{ll}_\lambda)^2 = \pi(f^{ll}_\lambda \times f^{ll}_\lambda) = \pi(f^{ll}_\lambda)$ we find that $\pi(f^{ll}_\lambda)$ equals either 0 or 1. If we assume the latter for a fixed value k of l then we have for $l \neq k$:

$$\pi(f^{ll}_\lambda) = \pi(f^{kk}_\lambda)\pi(f^{ll}_\lambda) = \pi(f^{kk}_\lambda \times f^{ll}_\lambda) = \delta_{kl}\pi(f^{ll}_\lambda) = 0. \tag{A.33}$$

Note that since the unit of $F^{*\,\mathrm{diag}}_\lambda$ must be respected $\pi(f^{ll}_\lambda)$ cannot vanish for all l. The irreducible representations of $F^{*\,\mathrm{diag}}_\lambda$ are thus given by:

$$\pi_l : f^{l'l'}_\lambda \mapsto \delta_{ll'}. \tag{A.34}$$

Any non-trivial irreducible representation (π, V) of F^*_λ can be decomposed into a sum of irreducible representations of $F^{*\,\mathrm{diag}}_\lambda$. Hence there is a $v^k \in V$ such that $\pi(f^{ll}_\lambda)v^k = \delta_{lk}v^k$. Let us define a set of n vectors in V by $v^m = \pi(f^{mk}_\lambda)v^k$. The span of $\{v^m\}$ defines an invariant subspace of V. This follows again from the product rule:

$$\begin{aligned}\pi(f^{ll'}_\lambda)v^m &= \pi(f^{ll'}_\lambda)\pi(f^{mk}_\lambda)v^k = \pi(f^{ll'}_\lambda \times f^{mk}_\lambda)v^k \\ &= \delta_{l'm}\pi(f^{lk}_\lambda)v^k = \delta_{l'm}v^l.\end{aligned} \tag{A.35}$$

Since π is irreducible the span of $\{v^m\}$ is V.

The claim is that V is n-dimensional. In order to prove this we have to show that the vectors v^m are linearly independent. If

$$\sum_m a_m v^m = 0 \tag{A.36}$$

one finds from (A.35):

$$f_\lambda^{ll}\left(\sum_m a_m v^m\right) = a_l v^l = 0. \tag{A.37}$$

So either $a_l = 0$ or $v^l = 0$. However, $v^l = 0$ together with the product rule and the definition of v^m implies that $v^m = \pi(f_\lambda^{ml})v^l = 0$. This would mean that $V = \{0\}$ contradicting the fact that V is non-trivial i.e. at least one dimensional. We thus find that an irreducible representation of F_λ^* is n-dimensional and moreover it follows from the explicit action on a basis of V as in equation (A.35) that such an irreducible representation is unique up to isomorphy.

We have found that an irreducible representation of F^* is completely fixed by a dominant integral weight λ in the appropriate weight lattice. The dimension of such an irreducible representation is given by the dimension of the irreducible representation of H^* with highest weight λ. To find the fusion rules for these representations we go back to the representations $\{\pi^\lambda\}$ introduced in formula (A.25) and (A.26) via the matrix entries of the original H^*-representations. The dimensions of these representations are given by the dimensions of the corresponding highest weight representations of H^*. Moreover they satisfy $\pi^\lambda(f_\mu^{ll'}) = 0$ for $\mu \neq \lambda$, i.e. π^λ defines a representation of F_λ^*. By comparing (A.26) and (A.35) one finds that π^λ corresponds precisely to the unique non-trivial irreducible representation of F_λ^*. We conclude that the representations $\{\pi^\lambda\}$ are the irreducible representations of F^*. Since the labels, the matrix elements and hence also the dimensions of these irreducible representations match those of the irreducible representations of H^* is seems very likely that the fusion rules for these representations of F^* are also identical to the fusion rules of the corresponding H^*-representations.

We have seen that the representation of $F^*(H^*)$ are identical to the representations of H^*. One might thus wonder to what extent H^* and $F^*(H)$ are equivalent. If H^* is a finite group one would find that $F^*(H^*)$ being a double dual of $\mathbb{C}H^*$ is isomorphic to the group algebra $\mathbb{C}H^*$. Since in our cases H^* is a continuous group one has to take care in taking the dual. Nonetheless one can define $F(H^*)$ as the dual of H^* via the irreducible representations of H^*. Similarly, one can retrieve H^* from the co-representations of $F^*(H)$. These co-representations are nothing but the representations of $F(H)$ which is the dual of $F^*(H)$. Let us illustrate this for $H^* = U(1)$.
An irreducible representation of $U(1)$ is uniquely labelled by an integer number. It is not very hard to check from equation (A.7) that the product of $F(U(1))$ can be expressed as:

$$\pi^n \times \pi^{n'} = \pi^{n+n'}. \tag{A.38}$$

Since $F(U(1))$ is commutative its irreducible representations are 1-dimensional. An irreducible representation thus sends π^1 to some $z \in \mathbb{C}$. It follows from (A.38) that the representation is completely defined by z:

$$z : \pi^n \mapsto z^n \in \mathbb{C}. \tag{A.39}$$

Not all values of z give a representation of $F(U(1))$ though. To give an example we note that π^{-1} is mapped to z^{-1}. This goes wrong for $z = 0$. For each $z \in \mathbb{C}\backslash\{0\}$ one does find a proper representation. It is easy to check that $\mathbb{C}\backslash\{0\}$ is a group. Obviously this is not the group $U(1)$. As matter of fact we have reconstructed the complexification $U(1)_{\mathbb{C}}$ of $U(1)$. To understand this we note that $U(1)$ has an involution which takes $h \mapsto h^* = h^{-1}$. The representations of $U(1)$ respect this involution in the sense that $(\pi^n(h))^* = \pi^n(h^*)$. We therefore have a natural involution on $F^*(U(1))$ defined by $(\pi^n)^* = \pi^{-n}$. Again one can define the representations of $F(U(1))$ to respect the involution, i.e. $(z(\pi^n))^* = z((\pi^n)^*)$. This results in the condition $z^* = z^{-1}$ which restricts z to the unit circle in $\mathbb{C}$, i.e. to $U(1)$.

For $SU(2)$ broken to $U(1)$ the Murray cone is the set of all non-negative integers. This implies that the dual $F_+(U(1))$ of $F_+^*(U(1))$ is generated by $\{\pi^n : n \geq 0\}$. The product is still given by (A.38) and hence $F_+(U(1))$ is a commutative algebra. Again we define an irreducible representation by $\pi^1 \mapsto z \in \mathbb{C}$. Since the representation should respect the product we find that the choice of z completely fixes the representation, i.e. $z : \pi^n \mapsto z^n$. One might again wonder if all values of z give a representation. Note that $F_+(U(1))$ is not closed under inversion just as the Murray cone is not closed under inversion. For example $\pi^{-1} \notin F_+(U(1))$. The representation labelled by $z = 0$ is thus not immediately ruled out. Note that for $z = 0$ we have $\pi^n \mapsto 0$ for all $n > 0$. The image of π^0 seems undetermined, nonetheless we can set $z(\pi^0) = z_0 \in \mathbb{C}$ for $z = 0$. The representation we now obtain does respect the product if and only if z_0 equals either 0 or 1. But since π^0 is the unit of the algebra it should be mapped to the unit of $\mathbb{C}$. Hence we find that $z(\pi^0) = 1$ for all $z \in \mathbb{C}$ and in particular for $z = 0$.
We have found that we should identify $U(1)_+$ with $\mathbb{C}$. The complex numbers are indeed closed under multiplication and moreover this multiplication is associative. On the other hand there is no inverse. We thus see that $U(1)_+$ is a semi-group and not a group as $U(1)$. Let us finally connect both ends of the circle and see if the commutative algebra $U(1)_+ = \mathbb{C}$ has the appropriate irreducible representations. Obviously $U(1)_+$ has representations π^n for $n > 0$ defined by:

$$\pi^n : z \mapsto z^n. \tag{A.40}$$

Representations with $n < 0$ do not exist because the image of $z = 0$ would not be defined. Finally the representation π^0 is a bit tricky. One can, however, simply define $\pi^0(0) = z_0$. It follows from the product on $\mathbb{C}$ that z_0 equals either 0 or 1. If, however, we restrict all representations to be continuous we find $z_0 = 1$. It is now almost trivial to check that the fusion rules of $U(1)_+$ correspond precisely to the fusion rules of $U(1)$. It would be interesting to study if a smooth semi-group H_+^* can be defined for every possible residual dual gauge group H^*.

APPENDIX B

WEYL GROUPS

In this appendix we first review and list the Weyl groups of the classical groups. For some simple examples we consider their irreducible representations. For a more detailed discussion we refer to e.g. [97]. These are used in appendix C to work out some examples of proto skeleton group representations. The characters of the Weyl group representations can be used to determine the fusion rules for the related proto skeleton groups.

B.1 WEYL GROUPS OF CLASSICAL LIE ALGEBRAS

The Weyl group is generated by reflections in the hyperplanes orthogonal to the roots and thus consist only of orthogonal transformations. This is why the structure of the Weyl group becomes particularly evident when an orthonormal basis of the weight space is used. To streamline matters even more one can put an additional requirement on such a basis, namely that the coordinates of any root are integers between -2 and 2. Such a basis exists for every classical Lie algebra except $A_r = \mathfrak{su}(r+1)$. However in this case it is still possible to accommodate this requirement by choosing an embedding of the weight space in $\mathbb{R}^{r+1}$. Thus will of course not give the familiar roots of $\mathfrak{su}(r+1)$ in $\mathbb{R}^r$, but this more unusual embedding is particular convenient for deriving the Weyl group. In table B.1 below we have listed these roots in terms of these bases for the classical groups, while in figure B.1 we have drawn the root diagrams for the simplest cases. For these examples it is quite simple to find the Weyl groups, and using the roots as expressed in table B.1 it does not require much more effort to generalise to any rank. First for $\mathfrak{su}(r+1)$ we see that the fundamental reflection w_i in the hyperplane orthogonal to the simple root

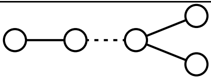
$\mathfrak{g}$	Dynkin diagram	simple roots	positive roots
A_r		$e_i - e_{i+1} \quad 1 \le i \le r$	$e_i - e_j \quad 1 \le i < j \le r{+}1$
B_r		$e_i - e_{i+1} \quad 1 \le i \le r{-}1$ e_r	$e_i \pm e_j \quad 1 \le i < j \le r$ $e_i \quad 1 \le i \le r$
C_r		$e_i - e_{i+1} \quad 1 \le i \le r{-}1$ $2e_r$	$e_i \pm e_j \quad 1 \le i < j \le r$ $2e_i \quad 1 \le i \le r$
D_r		$e_i - e_{i+1} \quad 1 \le i \le r{-}1$ $e_{r-1} + e_r$	$e_i \pm e_j \quad 1 \le i < j \le r$

Table B.1: *Roots in the orthonormal basis*

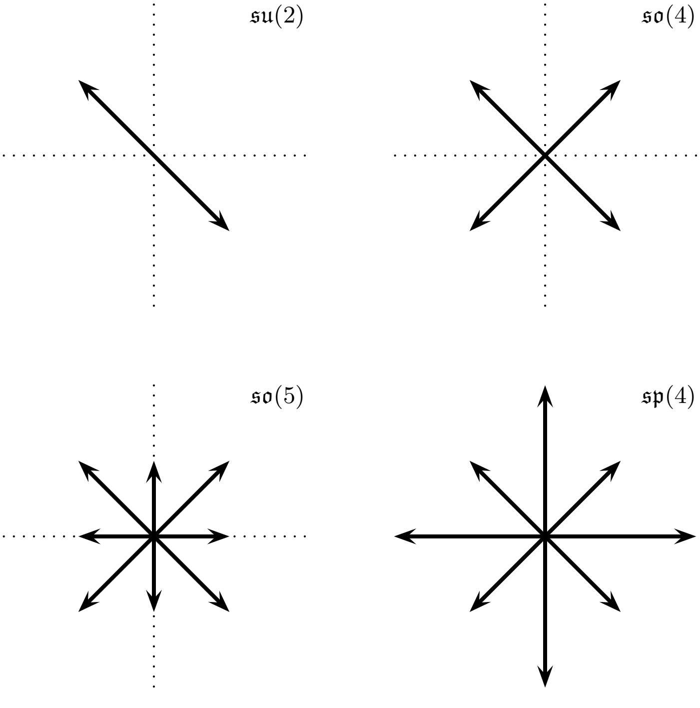


Figure B.1: *Root diagrams in the orthonormal basis*

$\alpha_i = e_i - e_{i+1}$ acts on the orthonormal basis as

$$w_{(i)} : \begin{cases} e_i \leftrightarrow e_{i+1} \\ e_j \mapsto e_j \quad \text{for } j \neq i, i+1. \end{cases} \tag{B.1}$$

In other words the fundamental reflections w_i interchange the coordinates of a weight

$$w_i : \lambda = (\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_{r+1}) \mapsto (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_{r+1}) \tag{B.2}$$

and thereby generate all possible permutations of the $r+1$ coordinates. We thus see that the orthonormal basis is convenient to derive the commonly know fact that $\mathcal{W}(\mathfrak{su}(n)) = \mathcal{S}_n$. For the other classical Lie groups, the first $r-1$ fundamental reflections again generate all permutations of the r coordinates of the weights. The reflection in the hyperplane orthogonal to the rth root however also involves a sign change. For both $\mathfrak{sp}(2r)$ and $\mathfrak{so}(2r+1)$ we have

$$w_r : \lambda = (\lambda_1, \ldots, \lambda_r) \mapsto (\lambda_1, \ldots, -\lambda_r), \tag{B.3}$$

While the rth fundamental reflection in $\mathcal{W}(\mathfrak{so}(2r)$ we actually have two sign flips since

$$w_r : \lambda = (\lambda_1, \ldots, \lambda_{r-1}, \lambda_r) \mapsto (\lambda_1, \ldots, , -\lambda_{r-1}, -\lambda_r). \tag{B.4}$$

We conclude that the Weyl groups of these three Lie algebras act by permuting the coordinates in the orthonormal basis and multiplying them by signs. For $\mathcal{W}(\mathfrak{so}(2r)$ we have the additional condition that only an even number of sign flips is allowed. The transformations with only a single sign change are actually the symmetries of the roots system induced by the $\mathbb{Z}_2$ symmetry of the Dynkin diagram.

It is very important to note that the permutations and the sign flips are not independent. Pick any element w in the Weyl group that only changes signs: $w = (s_1, \ldots, s_r)$ with $s_i \in \mathbb{Z}_2$. Conjugation of this group element by a pure permutation π returns again a pure sign flip element. But the positions where the sign flips occur have been permuted:

$$\pi(s_1, \ldots, s_r)\pi^{-1} = (s_{\pi(1)}, \ldots, s_{\pi(r)}). \tag{B.5}$$

From these considerations it follows that the Weyl groups of $\mathfrak{sp}(2r)$, $\mathfrak{so}(2r+1)$ and $\mathfrak{so}(2r)$ have the structure of a semi-direct product as given in the table below.[1]

$\mathfrak{g}$	$\mathcal{W}$
$\mathfrak{su}(n)$	$\mathcal{S}_n$
$\mathfrak{so}(2r+1)$	$\mathcal{S}_r \ltimes \mathbb{Z}_2^r.$
$\mathfrak{sp}(2r)$	$\mathcal{S}_r \ltimes \mathbb{Z}_2^r$
$\mathfrak{so}(2r)$	$\mathcal{S}_r \ltimes \mathbb{Z}_2^{r-1}$

[1]Note that there are exceptions. Obviously the Weyl group of $\mathfrak{so}(3) = \mathfrak{su}(2)$ is not a semi-direct product. For $\mathfrak{so}(4) = \mathfrak{su}(2) \times \mathfrak{su}(2)$ the action of the permutation group is also trivial.

B.2 Representations of the Weyl group

The Weyl group of $SU(n)$ is the symmetric group $\mathcal{S}_n$. The irreducible representations of the symmetric group are well known. The simplest non trivial case is $\mathcal{S}_2 \approx \mathbb{Z}_2$. This group has two 1-dimensional irreducible representations. We shall denote the trivial representation by Π_0. For the non trivial representation we shall write Π_1. Computing tensor products of $\mathbb{Z}_2$ representations is very easy: $\Pi_1 \otimes \Pi_1 = \Pi_0$. Besides S_2 we shall be using S_3 in some of the examples in the upcoming sections. Therefore we have summarised some facts basic facts in the character tables of these groups below.

$\mathcal{S}_2$	Π_0 □□	Π_1 □ □
e	1	1
w_1	1	-1

$\mathcal{S}_3$	Π_0 □□□	Π_1 □ □ □	Π_2 □□ □
$\{e\}$	1	1	2
$\{w_1, w_2, w_3\}$	1	-1	0
$\{w_1 w_2, w_2 w_3\}$	1	1	-1

Except in the case of $SU(n)$ the Weyl groups of classical groups are semi-direct products of the symmetric group and a normal subgroup that equals some power of $\mathbb{Z}_2$. Hence to find their irreducible representations one can use the method of induced representations as reviewed in section 4.4.1. In practice this means that one can assign either a trivial or a non trivial representation to each $\mathbb{Z}_2$ factor. These factors are permuted by the action of the symmetric group and their representations are interchanged accordingly. Hence the representation of the Weyl group will have an additional charge corresponding to the subgroup of permutations that do not interchange trivial and non-trivial $\mathbb{Z}_2$ representations.[2]

We finish this section with an example which we will be using later on. We shall compute the character table of $\mathcal{W}(Sp(4)) = \mathcal{S}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. This Weyl groups is actually the dihedral group D_4, the symmetry group of the square spanned by the fundamental representation of $Sp(4)$, see figure B.2.[3] This group has 8 elements which we denote by $(\pi, s_1, s_2) \in \mathcal{S}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The semi-direct product structure shows op in the multiplication of 2 elements: $(\pi, s_1, s_2)(\pi', s_1', s_2') = (\pi\pi', s_1 s'_{\pi(1)}, s_2 s'_{\pi(2)})$.
The 8 group elements fall into 5 different conjugacy classes which can be represented by the following group elements: $e = (1,1,1),\ w_1 = (1,-1,1),\ w_2 = (-1,1,1), w_1 w_2 = (-1,-1,1)$ and finally $w_1 w_2 w_1 w_2 = (1,-1,-1)$, where w_1 and w_2 are the reflections

[2]Since the Weyl group of $SO(2r)$ has has one $\mathbb{Z}_2$ factor less, its irreducible representations will have quantum numbers corresponding to some subgroup of $\mathcal{S}_{r-1}$.

[3]Notice the change in ordering of the simple roots with respect to table B.1.

in the axes perpendicular to α_1 and α_2. $w_1 w_2$ is a rotation over $\pi/2$. Consequently D_4 has 5 irreducible representations.
Four of these irreducible representations are quite easy to find. They correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2$ charges that are left invariant by the $\mathcal{S}_2$ action, i.e. these representations have a quantum number corresponding to the full $\mathcal{S}_2$ group. Thereby we find the following one-dimensional representations:

$$\Pi_{(i_1,i_2,i_2)} : (\pi, s_1, s_2) \mapsto \Pi_{i_1}(\pi)\Pi_{i_2}(s_1)\Pi_{i_2}(s_2). \tag{B.6}$$

Two remarks should be made. First note that these representations are invariant under the $\mathcal{S}_2$ action, they do not depend on the order of s_1 and s_2. It is therefore trivial that these representations respect the group multiplication of the semi-direct product. Second, the characters for these representations can be computed simply by counting signs, see the table below.

To find the last irreducible representation of D_4 one starts with $\mathbb{Z}_2$ charges $(i_1, i_2) = (1, 0)$. First note this pair is only invariant under the action of the trivial element in $\mathcal{S}_2$. This implies that we have an irreducible representation without an additional $\mathcal{S}_2$, which we shall denote by $\Pi_{(1,0)}$. Furthermore it is good to remember that the representations with charges in an orbit of $\mathcal{S}_2$ are equivalent, e.g. $\Pi_{(1,0)} = \Pi_{(0,1)}$.
To find the characters of this last irreducible representation one can use equation (4.35) with $\sigma_\pi \in \Pi_4/(\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathcal{S}_2$. Since $(\pi, s_1, s_2)\sigma_{\pi'} = \sigma_{\pi\pi'}$, we find

$$\chi((\pi, s_1, s_2)) = \begin{cases} 0 & \text{for } \pi \neq e \\ \sum_{\pi'} \chi(\pi'^{-1}(s_1, s_2)\pi') = s_1 + s_2 & \text{for } \pi = e. \end{cases} \tag{B.7}$$

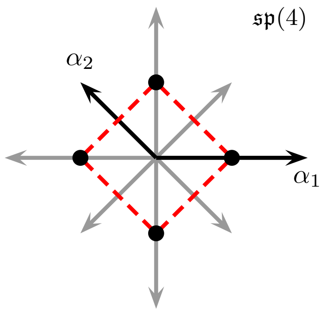


Figure B.2: *The fundamental representation of $Sp(4)$.*

$\mathcal{S}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	$\Pi_{(0,0,0)}$	$\Pi_{(1,0,0)}$	$\Pi_{(0,1,1)}$	$\Pi_{(1,1,1)}$	$\Pi_{(1,0)}$
$(+1, +1, +1)$	1	1	1	1	2
$(+1, \pm 1, \mp 1)$	1	1	-1	-1	0
$(-1, \pm 1, \pm 1)$	1	-1	1	-1	0
$(-1, \pm 1, \mp 1)$	1	-1	-1	1	0
$(+1, -1, -1)$	1	1	1	1	-2

Table B.2: *Character table of the Weyl group of* $\mathfrak{sp}(4)$ *and* $\mathfrak{so}(5)$.

APPENDIX C

PROTO SKELETON GROUP FOR CLASSICAL LIE GROUPS

Below the representation theory is considered fro the proto skeleton group associated to the groups $SU(2)$, $SU(3)$ and $Sp(4)$.

C.1 PROTO SKELETON GROUP FOR SU(2)

In this section we compute the irreducible representations and the characters of the proto skeleton group $\mathcal{W} \ltimes (T \times T^*)$ starting out from a Yang-Mills theory with $G = SU(2)$. The Weyl group of $SU(2)$ is of course $\mathbb{Z}_2$. The maximal torus T of $SU(2)$ is simply the subgroup $U(1) \subset SU(2)$ generated by $H_\alpha = \sigma_3$. The dual group of $SU(2)$ is $SO(3) = SU(2)/\mathbb{Z}_2$. We thus find that T^* can be identified with $U(1)/\mathbb{Z}_2$ where $U(1)$ is generated by σ_3 and $\mathbb{Z}_2$ is generated by $-\mathbb{I}$. We thus see that if we identify the electric weight lattice with the integer numbers the magnetic weight lattice is given by the even integers.
An irreducible representation of $T \times T^*$ is labelled by a pair $(2\lambda, 2g) \equiv (n, n^*) \in \mathbb{Z} \times \mathbb{Z}$ with n^* even. The Weyl group $\mathbb{Z}_2$ acts on these pairs as $(n, n^*) \mapsto (-n, -n^*)$. Hence only $(n, n^*) = (0, 0)$ has a non-trivial $\mathbb{Z}_2$-centraliser. In this case the centraliser actually equals $\mathbb{Z}_2$, and therefore the trivial charges together with the irreducible centraliser representation gives us a one-dimensional irreducible representation of $\mathbb{Z}_2 \ltimes (T \times T^*)$:

$$\Pi_{(i,[0,0])} : (w, g, \tilde{g}) \mapsto \Pi_i(w). \tag{C.1}$$

If either the electric or the magnetic charge is non-trivial we will obtain an irreducible representation of the proto skeleton group which is induced from the $\Pi_{(n,n^*)}$ representation of $T \times T^*$. As described in section 4.4.1 such an induced representation is constructed by using the action of the group on a coset space. In this case the coset space is the proto skeleton group modded out by $T \times T^*$, which is isomorphic to the Weyl group $\mathbb{Z}_2$. Since there are two cosets the induced representation is two dimensional. We shall denote these representations by $\Pi_{[n,n^*]}$.
Before we continue to discuss the tensor products of these irreducible representations one last remark should be made: Irreducible representations with charges related by the action of the Weyl group are equivalent, i.e. $\Pi_{[n,n^*]} = \Pi_{[-n,-n^*]}$. For the pure electric representations with $n^* = 0$ this means that the representation is defined unambiguously by the absolute value $|n|$.

The fusion rules for the proto skeleton group can be computed by evaluating equation (4.42). This gives the following results. If the charges are zero one simply retrieves the $\mathbb{Z}_2$ fusion rules. For nonzero charges one finds

$$(i, [0,0]) \otimes [n, n^*]) = [n, n^*] \tag{C.2}$$

$$[n_1, n_1^*] \otimes [n_2, n_2^*] = [n_1 + n_2, n_1^* + n_2^*] \oplus [n_1 - n_2, n_1^* - n_2^*]. \tag{C.3}$$

If the charges are equal (up to a Weyl transformation) this fusion rule is slightly different

$$[n, n^*] \otimes [n, n^*] = (0, [0,0]) \oplus (1, [0,0]) \oplus [2n, 2n^*]. \tag{C.4}$$

These fusion rules can be understood from two perspectives. First, if we take either the magnetic or the electric charges zero we obtain the fusion rules of $O(2)$. This should not be surprising at all since $\mathbb{Z}_2 \ltimes U(1) \cong O(2)$. Second, the fusion rules above are nothing but a logical extension and respect the fusion rules of $\mathbb{Z}[\Lambda \times \Lambda^*]$ as discussed in section 4.4.4.

C.2 PROTO SKELETON GROUP FOR SU(3)

The next example we are going to work out corresponds to an $SU(3)$ theory. All that we shall do here is repeating the recipe from the previous section. Nonetheless, the Weyl group of $SU(3)$ is truly non abelian and therefore the representation theory of the related proto skeleton group is potentially much more interesting. By the same token it is also much more complicated. We shall still be able to work out all irreducible representations. We shall deal with the fusion rules on the other hand on a case by case basis.

To avoid too much cluttering we shall use a compact notation and use $\mathcal{T}$ to denote $T \times T^*$. Thus for the proto skeleton group we write $\mathcal{S}_3 \ltimes \mathcal{T}$. As before we should start by choosing the charges corresponding to the $U(1)$ factors in $\mathcal{T}$, and determine the centraliser subgroup. The $\mathcal{T}$ charge which we denote by μ has 2 components, one related to the electric charge and one to the magnetic charge. Each of these components correspond to a point in the weight lattice of $SU(3)$, the magnetic charge however is restricted to the root lattice. The Weyl group acts on the charge μ. To visualise this action one take 2 copies of the $SU(3)$ weight lattice. The Weyl group acts on these charges simultaneously. It follows that the centraliser of μ is simply the intersection of the electric and the magnetic centralisers. We distinguish 3 different classes of Weyl orbits, with either 1, 3 or 6 elements, corresponding to 3 different classes of representations of the proto skeleton group.

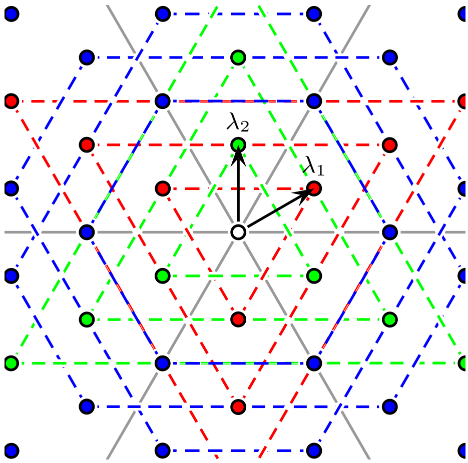


Figure C.1: *Weyl orbits in the weight lattice of* $SU(3)$.

The simplest case is when the Weyl orbit is trivial, that is when the charges are zero. All elements in the Weyl group leave this weight fixed which means that the centraliser subgroup is $\mathcal{S}_3$ itself. Thus by choosing a $\mathcal{S}_3$-representation Π_i we define a representation of the complete proto skeleton group:

$$\Pi_{(i,[0])} : (w, t) \mapsto \Pi_i(w). \tag{C.5}$$

If the charge is a multiple of either (λ_1, λ_1) or (λ_2, λ_2) the centraliser subgroup is isomorphic to $\mathbb{Z}_2 \subset \mathcal{S}_3$. Choosing a charge corresponding to an irreducible representation of the $\mathbb{Z}_2$ centraliser group gives us a one dimensional representation of $\mathbb{Z}_2 \ltimes \mathcal{T}$. From the fact that in these cases the Weyl orbits of the charges have 3 elements we conclude that the induced represention $\Pi_{(i,[\mu_k])}$ are 3 dimensional. Finally there is a set of charges with

trivial centralisers. These lead to six dimensional representations of the proto skeleton group denoted by $\Pi_{[\mu]}$.
We shall finally compute some fusion rules for the skeleton group of $SU(3)$. If we restrict to either pure electric case charges we have $\mu = n\lambda_1 + m\lambda_2$ for some positive integers n and m. Here we use the fundamental weights with $2\lambda_i \cdot \alpha_j/\alpha_j^2 = \delta_{ij}$. We shall first compute the fusion rule related to $3 \otimes 3$ in $SU(3)$.

$$(i, [\lambda_1]) \otimes (i, [\lambda_1]) = (0, [\lambda_2]) \oplus (1, [\lambda_2]) \oplus (0, [2\lambda_1) \tag{C.6}$$

$$(i, [\lambda_1]) \otimes (j, [\lambda_1]) = (0, [\lambda_2]) \oplus (1, [\lambda_2]) \oplus (1, [2\lambda_1]) \quad i \neq j. \tag{C.7}$$

As for as it concern the electric charges it is clear that this agrees with $3 \otimes 3 = 6 \otimes \overline{3}$ for $SU(3)$.

Some more fusion rules related to $3 \otimes \overline{3}$ are given by:

$$(i, [\lambda_1]) \otimes (i, [\lambda_2]) = [\lambda_1 + \lambda_2] \oplus (0, [0]) \oplus (2, [0]) \tag{C.8}$$

$$(i, [\lambda_1]) \otimes (j, [\lambda_2]) = [\lambda_1 + \lambda_2] \oplus (1, [0]) \oplus (2, [0]) \quad i \neq j. \tag{C.9}$$

If we ignore the centraliser charges this corresponds to $3 \otimes \overline{3} = 8 \oplus 1$.

C.3 PROTO SKELETON GROUP FOR SP(4)

In the last case we work out in considerable detail we take the residual gauge symmetry to contain a factor $Sp(4)$. This example is not much different from the $SU(3)$ case, except for the fact that $Sp(4)$ is not selfdual. Consequently the magnetic lattice is not directly embedded in the electric weight lattice but corresponds to the weight lattice of the dual group. Once we have taken this into account we can follow exactly the same procedure as before to compute the irreducible representations and the fusion rules.

As in the previous sections we shall denote the proto skeleton group by $\mathcal{W} \ltimes \mathcal{T}$. Where $\mathcal{T}$ contains the maximal tori of both the electric group $Sp(4)$ and its magnetic dual $SO(5)$. The Weyl group in this particular case is $D_4 = \mathcal{S}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$ as discussed in appendix B.1 and B.2. To construct the irreducible representations we first choose the $\mathcal{T}$ charge $\mu = (\lambda, g)$. The centraliser of μ is either $D_4 \times \mathcal{T}$, $\mathbb{Z}_2 \ltimes \mathcal{T}$ or $\mathcal{T}$. We shall discuss these cases separately.
Only if μ is zero it is invariant under the whole Weyl group. This will lead to irreducible representations $\Pi_{(i,[0])}$ which correspond to representations Π_i of D_4 reviewed in appendix B.2.
There are several possibilities to realise a $\mathbb{Z}_2$ centraliser. In each case the subgroup

$\mathbb{Z}_2 \subset \mathcal{W}$ is generated by the refection w_α that leaves the $\mathcal{T}$-charge fixed. Taking α to be a simple root is sufficient to capture all the isomorphism classes of irreducible representations. This leaves us with two possibilities corresponding to α_1 and α_2 as depicted in figure B.2. In the first case the centraliser corresponds to a sign flip in the second case the $\mathbb{Z}_2$ subgroup comes from the permutation of the coordinates of the weight space. Either way the resulting Weyl orbits have four elements and since there are two irreducible $\mathbb{Z}_2$-representations one obtains two inequivalent 4-dimensional proto skeletongroup representations for each such Weyl orbit.
Finally one can have an orbits represented by $\mu \in \Lambda \times \Lambda^*$ such that μ has only a trivial centraliser. Such orbits thus have 8 elements and corresponding irreducible representation $\Pi_{[\mu]}$ which are 8 dimensional.

Fusion rules for the skeleton group of $Sp(4)$ can be computed from formula (4.42) using the characters of D_4 listed in table B.2.

APPENDIX D

SKELETON GROUP FOR CLASSICAL LIE GROUPS

A subtle part in constructing the skeleton group is determining the lift of the Weyl group to Lie group. The main part of this appendix is therefore dedicated to describing these lifts for the classical groups. We shall also determine the relevant normal groups that by modding out give the Weyl group back.

D.1 SKELETON GROUP FOR SU(N)

Below we work out the construction of the skeleton group and its irreducible representations in some detail for $G = SU(n)$.

We shall start by identifying the lift W of the Weyl group. For the maximal torus T of $SU(n)$, we take the subgroup of diagonal matrices. The length of the roots is set to $\sqrt{2}$. The raising and lowering operators for the simple roots are the matrices given by $(E_{\alpha_i})_{lm} = \delta_{li}\delta_{m,i+1}$ and $(E_{-\alpha_i})_{lm} = \delta_{l,i+1}\delta_{m,i}$. From this one finds that x_{α_i} as defined in equation (4.26) is given by:

$$(x_{\alpha_i})_{lm} = \delta_{lm}(1 - \delta_{li} - \delta_{l,i+1}) + i(\delta_{li}\delta_{m,i+1} + \delta_{l,i+1}\delta_{mi}). \tag{D.1}$$

From now on we abbreviate x_{α_i} to x_i. One easily shows that

$$x_i^4 = 1, \qquad [x_i, x_j] = 0 \text{ for } |i - j| > 1, \qquad x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}. \tag{D.2}$$

As it stands, this is not the complete set of relations for W. However, one may show that W is fully determined if we add the relations

$$(x_i x_{i+1})^3 = 1. \tag{D.3}$$

This also makes contact with the presentation of the normaliser of T obtained by Tits [98, 99].

We shall now determine the group D. Note that the elements $x_i^2 \in W$ are diagonal and of order 2. In fact we have $(x_i^2)_{lm} = \delta_{lm}(1 - 2\delta_{li} - 2\delta_{l,i+1})$. One thus sees that the group K generated by the x_i^2 is just the group of diagonal matrices with determinant 1 and diagonal entries equal to ± 1. Since its elements are diagonal we have $K \subset T$ and hence $K \subset D = W \cap T$. As a matter of fact $K = D$. To prove this one can check that conjugation with the x_i leaves K invariant. Hence K is a normal subgroup of W and thus the kernel of some homomorphism ρ on W. The image of ρ is the Weyl group $\mathcal{S}_n$ of $SU(n)$. To see this note that W/K satisfies the relations of the permutation group (these are the same as the relations for the x_i above, but with $x_i^2 = 1$). An explicit realisation of $\rho : W \to \mathcal{S}_n$ is given by $\rho(w) : t \in T \mapsto wtw^{-1}$. Obviously $D \subset \text{Ker}(\rho) = K$, consequently $D = K$.

Let us work out the $SU(2)$ case as small example. $SU(2)$ has only a simple root and thus W as only one generator x which satisfies $x^4 = 1$. This gives $W = \mathbb{Z}_4$. D is generated by x^2 which squares to the indentity and hence $D = \mathbb{Z}_2$. For higher rank W is slightly more complicated but D is simply given by the abelian group $\mathbb{Z}_2^{n-1}$.

In order to determine the representations of S for $SU(n)$ we need to solve (4.52) and hence we need to describe how D is represented on a state $|\,\lambda\,\rangle$ in an arbitrary representation of $SU(n)$. This turns out to be surprisingly easy. The generating element x_i^2 of D acts as the non-trivial central element of the $SU(2)$ subgroup in $SU(n)$ that corresponds to α_i. Now let $(\lambda_1, \ldots, \lambda_{n-1})$ be the Dynkin labels of the weight λ. Note that λ_i is also the weight of λ with respect to the $SU(2)$ subgroup corresponding to α_i. Recall that the central element of $SU(2)$ is always trivially represented on states with an even weight while it acts as -1 on states with an odd weight. Hence x_i^2 leaves $|\,\lambda\,\rangle$ invariant if λ_i is even and sends $|\,\lambda\,\rangle$ to $\lambda(x_i^2)|\,\lambda\,\rangle = -|\,\lambda\,\rangle$ if λ_i is odd.

For any given orbit $[\lambda, g]$ we can solve (4.52) by determining $N_{\lambda,g} \subset W$ and choosing a representation of $N_{\lambda,g}$ which assures that the elements (x_i^2, x_i^2) act trivially on the vectors $|\,\lambda, v^\gamma\,\rangle$.
If the centraliser of $[\lambda, g]$ in $\mathcal{W}$ is trivial its centraliser $N_{(\lambda,g)}$ in W equals $D = \mathbb{Z}_2^{n-1}$. An irreducible representations of γ of D is 1-dimensional and satisfies $\gamma(x_i^2) = \pm 1$. The centraliser representations that satisfy the constraint (4.52) are defined by $\gamma(x_i^2) = \lambda(x_i^2)$. If $(\lambda, g) = (0, 0)$ such the centraliser is W. In this case an allowed centraliser representa-

tions γ satisfies $\gamma(d)|\,v\,\rangle = |\,v\,\rangle$, i.e. γ is a representation of $W/D = \mathcal{W}$. The irreducible representations $\Pi_\gamma^{[0,0]}$ of S thus correspond to irreducible representations of the permutation group $\mathcal{S}_n$.
If $N_{(\lambda,g)}$ is neither D nor W the situation is more complicated and we will not discuss this any further.

D.2 Skeleton group for Sp(2n)

In this section we shall consider the skeleton group for $Sp(2n)$. The skeleton group for the dual group $SO(2n+1)$ will be discussed in the next section.

In order to construct the lift W of the Weyl group to $Sp(2n)$ we need to define the Lie algebra, see e.g section 16.1 of [50]. The CSA is generated by $2n \times 2n$ matrices H_i defined by

$$(H_i)_{kl} = \delta_{ki}\delta_{li} - \delta_{k,n+i}\delta_{n+i,l}. \tag{D.4}$$

The short simple roots $\alpha_i = e_i - e_{i+1}$ with length $\sqrt{2}$ correspond to E_{α_i} and $E_{-\alpha_i}$ which are defined as

$$(E_{\alpha_i})_{kl} = \delta_{ki}\delta_{i+1,l} - \delta_{k,n+i+1}\delta_{n+i,l} \tag{D.5}$$

$$(E_{-\alpha_i})_{kl} = \delta_{k,i+1}\delta_{i,l} - \delta_{k,n+i}\delta_{n+i+1,l}. \tag{D.6}$$

The long simple root α_n with length 2 is related to the raising and lowering operator E_{α_n} and $E_{-\alpha_n}$ given by

$$(E_{\alpha_n})_{kl} = \delta_{kn}\delta_{2n,l} \tag{D.7}$$

$$(E_{-\alpha_i})_{kl} = \delta_{k,2n}\delta_{n,l}. \tag{D.8}$$

From this one finds that x_{α_i} as defined in equation (4.26) is given by:

$$\begin{aligned}(x_{\alpha_i})_{lm} \quad = \quad & \delta_{lm}(1 - \delta_{li} - \delta_{l,i+1} - \delta_{l,n+i} - \delta_{l,n+i+1}) + \\ & i(\delta_{li}\delta_{m,i+1} + \delta_{l,i+1}\delta_{mi} - \delta_{l,n+i}\delta_{m,n+i+1} - \delta_{l,n+i+1}\delta_{m,n+i}).\end{aligned} \tag{D.9}$$

While x_{α_n} is given by

$$(x_{\alpha_n})_{lm} = \delta_{lm}(1 - \delta_{ln} - \delta_{l,2n}) + i(\delta_{ln}\delta_{m,2n} + \delta_{l,2n}\delta_{mn}). \tag{D.10}$$

To avoid cluttering we abbreviate x_{α_i} to x_i and x_{α_n} to y_n. As follows from the results in section D.1 for $SU(n)$ the x_is generate a subgroup S_n of W which is completely defined by

$$x_i^4 = 1, \qquad [x_i, x_j] = 0 \text{ for } |i-j| > 1, \qquad x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \tag{D.11}$$

and

$$(x_i x_{i+1})^3 = 1. \tag{D.12}$$

Another subgroup of W is $\mathbb{Z}_4$ generated by y_n, which is nothing but the lift of $\mathbb{Z}_2 \in W$ generated by the Weyl reflection in the plane orthogonal to α_n. Note that $\mathcal{W}$ contains a subgroup $\mathbb{Z}_2^n$. One might thus expect that W contains a subgroup $\mathbb{Z}_4^n$. This is indeed the case. We define

$$(y_i)_{lm} = \delta_{lm}(1 - \delta_{li} - \delta_{l,n+i}) + i(\delta_{li}\delta_{m,n+i} + \delta_{l,n+i}\delta_{mi}). \tag{D.13}$$

Note that $y_i^4 = 1$ and $[y_i, y_j] = 0$. The y_is thus generate a subgroup $\mathbb{Z}_4^n$ in W. Moreover this subgroup is a normal subgroup of W since it is invariant under conjugation with all the generators of W. One can easilycheck that y_{i+1} is related to y_i via conjugation with x_i. One might thus expect that the lift of $\mathcal{W} = \mathcal{S}_n \ltimes \mathbb{Z}_2^n$ is simply $S_n \ltimes \mathbb{Z}_4^n$. Note however that $S_n \cap \mathbb{Z}_n^4 \neq \{e\}$. To determine the true value of this intersection one observes that $x_i^2 = y_i^2 y_{i+1}^2$. Consequently $S_n \cap \mathbb{Z}_n^4 = \mathbb{Z}_2^{n-1}$ generated by the x_i^2s and $W = (S_n \ltimes \mathbb{Z}_4^n)/\mathbb{Z}_2^{n-1}$.

Next we want to compute the intersection of W with the maximal torus T in $Sp(n)$. One immediately sees that y_i^2 is a diagonal matrix and thus an element of T. Since each y_i^2 generates a $\mathbb{Z}_2$ group one finds that $\mathbb{Z}_2^n \subset W \cap T$. Next we want to proof that $W \cap T \subset \mathbb{Z}_2^n$. We use the same approach as in the $SU(n)$ case. $\mathbb{Z}_2^n$ is a normal subgroup of W. Hence $\mathbb{Z}_2^n$ is the kernel of some homomorphism ρ. The image of this homomorphism is isomorphic to $W/\mathbb{Z}_2^n$. The defining relations of this group can be found from the defining relations of W and the equivalence relations $y_i^2 = 1$. Note that this equivalence relation also implies the relation $x_i^2 = 1$ and we thus retrieve the defining relations for $\mathcal{S}_n \ltimes \mathbb{Z}_2^n$. An explicit realisation of $\rho : W \to \mathcal{S}_n \ltimes \mathbb{Z}_n^2$ is given by $\rho(w) : t \in T \mapsto wtw^{-1}$. Obviously $W \cap T \subset \text{Ker}(\rho) = \mathbb{Z}_2^n$, consequently $W \cap T = \mathbb{Z}_2^n$.

D.3 Skeleton group for SO(2n+1)

Here we shall compute the skeleton group for $SO(2n+1)$ by determining the lift of the Weyl group W and its insertsection with the maximal torus.

In order to construct the lift W of the Weyl group to $SO(2n+1)$ we need to define the Lie algebra, see e.g. section 18.1 of [50]. The CSA is generated by $(2n+1) \times (2n+1)$ matrices H_i defined by

$$(H_i)_{kl} = \delta_{ki}\delta_{li} - \delta_{k,n+i}\delta_{n+i,l}. \tag{D.14}$$

The long simple roots α_i with length $\sqrt{2}$ correspond to E_{α_i} and $E_{-\alpha_i}$ which are defined as

$$(E_{\alpha_i})_{kl} = \delta_{ki}\delta_{i+1,l} - \delta_{k,n+i+1}\delta_{n+i,l} \tag{D.15}$$

$$(E_{-\alpha_i})_{kl} = \delta_{k,i+1}\delta_{i,l} - \delta_{k,n+i}\delta_{n+i+1,l}. \tag{D.16}$$

The short simple root α_n with length 1 is related to the raising and lowering operator E_{α_n} and $E_{-\alpha_n}$ which in our conventions are given by

$$(E_{\alpha_n})_{kl} = \sqrt{2}\,(\delta_{k,n}\delta_{2n+1,l} - \delta_{k,2n+1}\delta_{2n,l}) \tag{D.17}$$

$$(E_{-\alpha_n})_{kl} = \sqrt{2}\,(\delta_{k,2n+1}\delta_{n,l} - \delta_{k,2n}\delta_{2n+1,l})\,. \tag{D.18}$$

From this one finds that x_{α_i} as defined in equation (4.26) is given by:

$$\begin{aligned}(x_{\alpha_i})_{lm} \quad = \quad & \delta_{lm}(1 - \delta_{li} - \delta_{l,i+1} - \delta_{l,n+i} - \delta_{l,n+i+1}) + \\ & i(\delta_{li}\delta_{m,i+1} + \delta_{l,i+1}\delta_{mi} - \delta_{l,n+i}\delta_{m,n+i+1} - \delta_{l,n+i+1}\delta_{m,n+i}).\end{aligned} \tag{D.19}$$

While x_{α_n} is given by

$$(x_{\alpha_n})_{lm} = \delta_{lm}(1 - \delta_{l,n} - \delta_{l,2n} - 2\delta_{l,2n+1}) + (\delta_{ln}\delta_{m,2n} + \delta_{l,2n}\delta_{mn}). \tag{D.20}$$

To avoid cluttering we abbreviate x_{α_i} to x_i and x_{α_n} to y_n. As for $Sp(2n)$ we conclude that the x_is generate a subgroup S_n of W. In contrast to the $Sp(2n)$ case we have $y_n^2 = 1$, hence, instead of a $\mathbb{Z}_4$ subgroup y_n simply generates a $\mathbb{Z}_2$ subgroup. In addition we define the generators $y_i = x_i y_{i+1} x_i^{-1}$. Note that $y_i^4 = 1$ and $[y_i, y_j] = 0$. It should be clear that the y_is generate a normal subgroup of W which equals $\mathbb{Z}_2^n$. One can also check that this subgroup does not intersect with S_n except in 1. We thus find $W = S_n \ltimes \mathbb{Z}_2^n$.
Next we want to compute the intersection of W with the maximal torus T in $SO(2n+1)$. Note that y_i^2 equals the trivial element in T. On the other hand x_i^2 is a diagonal matrix not equal to the unit. By adapting our arguments from the $SU(n)$ and $Sp(2n)$ cases it should now be clear that $D = W \cap T$ is generated by the x_is and hence equals $\mathbb{Z}_2^{n-1}$. Moreover we indeed have that W/D equals the Weyl group $\mathcal{S}_n \ltimes \mathbb{Z}_2^n$ of $SO(2n+1)$.

D.4 Skeleton group for SO(2n)

Finally we shall determine the lift of the Weyl group for $SO(2n+1)$ and its intersection with the maximal torus. Together with the maximal torus itself and the dual torus this fixes the skeleton group.

As can be read off from table B.1 and as illustrated by figure B.1 the root diagram of $SO(2n)$ looks like the root diagrams of $SO(2n+1)$ and $Sp(2n)$ but with the exceptional roots removed. This similarity is directly reflected upon the matrix representations, see e.g. section 18.1 of [50]. The CSA of $SO(2n)$ is generated by $2n \times 2n$ matrices H_i defined by

$$(H_i)_{kl} = \delta_{ki}\delta_{li} - \delta_{k,2i}\delta_{2i,l}. \tag{D.21}$$

The first $n-1$ simple roots α_i with length $\sqrt{2}$ correspond to E_{α_i} and $E_{-\alpha_i}$ which are defined as

$$(E_{\alpha_i})_{kl} = \delta_{ki}\delta_{i+1,l} - \delta_{k,n+i+1}\delta_{n+i,l} \tag{D.22}$$

$$(E_{-\alpha_i})_{kl} = \delta_{k,i+1}\delta_{i,l} - \delta_{k,n+i}\delta_{n+i+1,l}. \tag{D.23}$$

The nth simple root α_n, whose $SO(2n+1)$ counterpart is actually not simple, is related to the raising and lowering operator E_{α_n} and $E_{-\alpha_n}$ given by

$$(E_{\alpha_n})_{kl} = \delta_{k,n-1}\delta_{2n,l} - \delta_{k,n}\delta_{2n-1,l} \tag{D.24}$$

$$(E_{-\alpha_n})_{kl} = \delta_{k,2n}\delta_{n-1,l} - \delta_{k,2n-1}\delta_{n,l}. \tag{D.25}$$

From this one finds that x_{α_i} as defined in equation (4.26) is given by:

$$\begin{aligned}(x_{\alpha_i})_{lm} &= \delta_{lm}(1 - \delta_{li} - \delta_{l,i+1} - \delta_{l,n+i} - \delta_{l,n+i+1}) + \\ &\quad i(\delta_{li}\delta_{m,i+1} + \delta_{l,i+1}\delta_{mi} - \delta_{l,n+i}\delta_{m,n+i+1} - \delta_{l,n+i+1}\delta_{m,n+i}).\end{aligned} \tag{D.26}$$

While x_{α_n} is given by

$$\begin{aligned}(x_{\alpha_n})_{lm} &= \delta_{lm}(1 - \delta_{l,n-1} - \delta_{l,n} - \delta_{l,2n-1} - \delta_{l,2n}) + \\ &\quad i(\delta_{l,n-1}\delta_{2n,m} + \delta_{l,2n}\delta_{n-1,m} - \delta_{l,n}\delta_{2n-1,m} - \delta_{l,2n-1}\delta_{n,m}).\end{aligned} \tag{D.27}$$

We abbreviate x_{α_i} to x_i with $i = 1, \cdots, n-1$. The x_is generate the group S_n. We define y_{n-1} as $x_{n-1}x_{\alpha_n}$ and finally $y_i = x_i y_{i+1} x_i^{-1}$. One can check that $y_i^2 = 1$ and $[y_i, y_j] = 0$ and in particular that the group Z_2^{n-1} generated by the y_is correspond to the double sign flips of the Weyl group action as discussed in appendix B.1. It is important to note that $\mathbb{Z}_2^{n-1}$ is invariant under the action of S_n. It is also not hard to see that $S_n \cap \mathbb{Z}_2^{n-1} = e$. The lift of the Weyl group W is thus given by $S_n \ltimes \mathbb{Z}_n^{n-1}$. Finally we note that $D = W \cap T$ equals $\mathbb{Z}_2^{n-1}$ generated by the set $\{x_i^2\}$.

APPENDIX E

GENERALISED TRANSFORMATION GROUP ALGEBRAS

We have seen that the irreducible representations of a semi-direct product are labelled by an orbit and a centraliser representation. This property is not unique. The irreducible representations of generalised transformations group algebras have a similar classification. This makes it interesting to consider a generalised transformation group algebra that reproduces the dyonic charge sectors as the skeleton group does. We shall see that such an generalised transformation group algebra also has the same fusion rules as the skeleton group. One thus might interpret such an algebra as a subalgebra of a unified electric-magnetic symmetry. A possible objection is that this transformation group algebra is not group and can thus not be interpreted as an invertible symmetry acting on states. On the other hand a transformation group algebra does contain a group algebra, which we can take to be the group algebra of an electric group.

E.1 IRREDUCIBLE REPRESENTATIONS

The definition of a general transformation group algebra can be found in [100], see also [101]. In the definition we will be using H is a finite group acting on a finite set Λ. Nonetheless there is a valid generalisation for H a locally compact group with a Haar measure and Λ a discrete set. $F(\Lambda \times H)$, the set of functions on $\Lambda \times H$, is called a transformation group algebra if it is equipped with a multiplication given by

$$f_1 f_2(\lambda, h) = \int_H f_1(\lambda, g) f_2(g^{-1} \triangleright \lambda, g^{-1} h) dg. \tag{E.1}$$

The unit of this algebra is given by a δ-function:

$$1 : (\lambda, h) \in \Lambda \times H \mapsto \delta_{e,h} \in \mathbb{C}, \tag{E.2}$$

where e is the unit of H.
If Λ is also a group then $F(\Lambda \times H)$ can be extended to a Hopf algebra. The coproduct, counit and antipode are defined by

$$\Delta(f)(\lambda_1, h_1; \lambda_2, h_2) = f(\lambda_1 \lambda_2, h_1) \delta_{h_1, h_2} \tag{E.3}$$

$$\epsilon(f) = \int_H f(e_\Lambda, h) dh \tag{E.4}$$

$$S(f)(\lambda, h) = f(h^{-1} \triangleright \lambda^{-1}, h^{-1}). \tag{E.5}$$

At least if both H and Λ are discrete sets there is a convenient basis for $F(\Lambda \times H)$ defined by the projection functions

$$P_\lambda h : (\mu, g) \in \Lambda \times H \mapsto \delta_{\lambda,\mu} \delta_{h,g}. \tag{E.6}$$

In terms of these projection operators the Hopf algebra is given by

$$P_{\lambda_1} h_1 P_{\lambda_2} h_2 = \delta_{\lambda_1, h_1 \triangleright \lambda_2} P_{\lambda_1} h_1 h_2 \tag{E.7}$$

$$\Delta(P_\lambda h) = \sum_{\mu \in \Lambda} P_{\lambda \mu^{-1}} h \otimes P_\mu h \tag{E.8}$$

$$1 = \sum_{\lambda \in \Lambda} P_\lambda h \tag{E.9}$$

$$\epsilon(P_\lambda h) = \delta_{e_\Lambda, \lambda} \tag{E.10}$$

$$S(P_\lambda h) = P_{h^{-1} \triangleright \lambda^{-1}} h^{-1}. \tag{E.11}$$

An intersting set of functions is given by

$$Ph : (\lambda, g) \in \Lambda \times H \mapsto \delta_{h,g} \in \mathbb{C}. \tag{E.12}$$

One can check that these function generate a subalgebra in $F(\Lambda \times H)$ which is isomorphic to the group algebra of H.

Before we review the the irreducible representations we introduce the Hilbert spaces upon which the representations will act. Let N be a subgroup of H, and γ a unitary representation of N acting on the space V_γ. We define a Hilbert space by the set of maps from H to V_γ that respect the action of N:

$$F_\gamma(H, V_\gamma) = \left\{ |\,\phi\,\rangle : H \to V_\gamma \mid |\,\phi\,\rangle(hn) = \gamma(n^{-1}) |\,\phi\,\rangle(h), \forall h \in H, \forall n \in N \right\}. \tag{E.13}$$

The irreducible representations of $F(\Lambda \times H)$ are described as follows. Let $[\lambda]$ be an orbit in Λ under the action of H and let N_λ be the centraliser of the representant $\lambda \in \Lambda$. Then, for each pair $([\lambda], \gamma)$ of an orbit $[\lambda]$ and γ an irreducible representation of N_λ, we have an irreducible unitary representation $\Pi^{[\lambda]}_\gamma$ of $F(\Lambda \times H)$ on $F_\gamma(H, V_\gamma)$ is given by

$$\Pi^{[\lambda]}_\gamma(f)|\,\phi\,\rangle(h) := \int_H f(h \triangleright \lambda, g)|\,\phi\,\rangle(g^{-1}h)dg. \tag{E.14}$$

Moreover, all unitary irreducible representations of $F(\Lambda \times H)$ are of this form and $\Pi^{[\lambda]}_\gamma$ and $\Pi^{[\mu]}_\alpha$ are equivalent if and only if $[\lambda] = [\mu]$ and $\gamma \cong \alpha$.

E.2 Matrix elements and characters

For an irreducible represention $\Pi^{[\lambda]}_\gamma$ we denote its carrier space $F_\gamma(H, V_\gamma)$ by $V^{[\lambda]}_\gamma$. The dimension of $V^{[\lambda]}_\gamma$ is equal to the product $d_{[\lambda]}$ of the number of elements in $[\lambda]$ and d_γ the dimension of V_γ. To see this note that the functions $|\,\phi\,\rangle \in V^{[\lambda]}_\gamma$ are completely determined once their value on one element in each coset hN_λ of H/N_λ is chosen. The number of cosets equals $d_{[\lambda]}$ while $\phi(h)$ has d_γ components. Hence $\dim V^{[\lambda]}_\gamma = d_{[\lambda]} d_\gamma$. To define a basis for $V^{[\lambda]}_\gamma$ we choose a basis $\{|\,e^\gamma_i\,\rangle\}$ for V_γ and $h_\mu \in H$ for each $\mu \in [\lambda]$ such that $h_\mu \triangleright \lambda = \mu$. The basis elements $\{|\,\mu; e^\gamma_i\,\rangle\}$ are given by the map

$$|\,\mu; e^\gamma_i\,\rangle : h_\nu n \in H \mapsto \delta_{\nu\mu}\gamma(n^{-1})|\,e^\gamma_i\,\rangle \in V_\gamma. \tag{E.15}$$

The action of $\Pi^{[\lambda]}_\gamma(f)$ can be found by evaluating equation (E.14) on h_ν for each $\nu \in [\lambda]$.

$$\begin{aligned}
\Pi^{[\lambda]}_\gamma(f)|\,\mu; e^\gamma_i\,\rangle(h_\nu) &= \int_H f(h_\nu \triangleright \lambda, g)|\,\mu; e^\gamma_i\,\rangle(g^{-1}h_\nu)dg \\
&= \sum_{\rho\in[\lambda]} \int_{N_\lambda} f(\nu, h_\nu n h_\rho^{-1})|\,\mu; e^\gamma_i\,\rangle(h_\rho n^{-1})dn \\
&= \sum_{\rho\in[\lambda]} \int_{N_\lambda} f(\nu, h_\nu n h_\rho^{-1})\delta_{\mu\rho}\gamma(n)|\,e^\gamma_i\,\rangle dn \\
&= \int_{N_\lambda} f(\nu, h_\nu n h_\mu^{-1})\gamma(n)|\,e^\gamma_i\,\rangle dn \\
&= \int_{N_\lambda} f(\nu, h_\nu n h_\mu^{-1})\gamma(n)_{ij}|\,e^\gamma_j\,\rangle dn \\
&= \int_{N_\lambda} f(\nu, h_\nu n h_\mu^{-1})\gamma(n)_{ij}dn|\,\nu; e^\gamma_j\,\rangle(h_\nu).
\end{aligned} \tag{E.16}$$

The matrix elements of $\Pi_\gamma^{[\lambda]}$ with respect to the basis $\{|\,\mu; e_i^\gamma\,\rangle\}$ are thus given by

$$\Pi_\gamma^{[\lambda]}(f)_{ij}^{\mu,\nu} = \int_{N_\lambda} f(\nu, h_\nu n h_\mu^{-1})\gamma(n)_{ij} dn. \tag{E.17}$$

In particular for $f = P_\sigma h$ we have

$$\begin{aligned}\Pi_\gamma^{[\lambda]}\left(P_\sigma h\right)_{ij}^{\mu,\nu} &= \int_{N_\lambda} P_\sigma h(\nu, h_\nu n h_\mu^{-1})\gamma(n)_{ij} dn \\ &= \int_{N_\lambda} \delta_{\sigma,\nu}\delta_{h,h_\nu n h_\mu^{-1}}\gamma(n)_{ij} dn \\ &= \delta_{\sigma,\nu}\delta_{h\triangleright\mu,\nu}\gamma(h_\nu^{-1} h h_\mu)_{ij}.\end{aligned} \tag{E.18}$$

Consequently the character $\chi_\gamma^{[\lambda]}$ of $\Pi_\gamma^{[\lambda]}$ is defined by

$$\chi_\gamma^{[\lambda]}(f) = \sum_{\mu\in[\lambda]} \int_{N_\lambda} f(\mu, h_\mu n h_\mu^{-1})\chi_\gamma(n) dn, \tag{E.19}$$

where χ_γ denotes the character of γ. For the projection operators this gives

$$\begin{aligned}\chi_\gamma^{[\lambda]}(P_\nu h) &= \sum_{\mu\in[\lambda]} \int_{N_\lambda} P_\nu h(\mu, h_\mu n h_\mu^{-1})\chi_\gamma(n) dn \\ &= \sum_{\mu\in[\lambda]} \int_{N_\lambda} \delta_{\nu\mu}\delta_{h,h_\mu n h_\mu^{-1}}\chi_\gamma(n) dn \\ &= \sum_{\mu\in[\lambda]} \delta_{\mu,\nu}\delta_{h\triangleright\nu,\nu}\chi_\gamma(h_\nu^{-1} h h_\nu).\end{aligned} \tag{E.20}$$

One may now define an inner product by the formula

$$\langle \chi_1, \chi_2 \rangle = \sum_{\lambda\in\Lambda} \int_H \chi_1(P_\lambda h)\chi_2^*(P_\lambda h) dh. \tag{E.21}$$

One may check that the characters of the irreducible representation are orthogonal with respect to this inner product:

$$\begin{aligned}\left\langle \chi_\gamma^{[\rho]}, \chi_\alpha^{[\sigma]} \right\rangle &= \sum_{\lambda\in\Lambda} \int_H \sum_{\mu\in[\rho]} \delta_{\mu,\lambda}\delta_{h\triangleright\lambda,\lambda}\chi_\gamma(h_\lambda^{-1} h h_\lambda) \sum_{\nu\in[\sigma]} \delta_{\nu,\lambda}\delta_{h\triangleright\lambda,\lambda}\chi_\alpha^*(h_\lambda^{-1} h h_\lambda) dh \\ &= \delta_{[\rho][\sigma]} \sum_{\mu\in[\rho]} \int_H \delta_{h\triangleright\mu,\mu}\chi_\gamma(h_\mu^{-1} h h_\mu)\chi_\alpha^*(h_\mu^{-1} h h_\mu) dh \\ &= \delta_{[\rho][\sigma]} \sum_{\mu\in[\rho]} \int_{N_\rho} \chi_\gamma(n)\chi_\alpha^*(n) dn \\ &= \delta_{[\rho][\sigma]}\delta_{\gamma\alpha} \sum_{\mu\in[\rho]} \int_{N_\rho} dn \\ &= \delta_{[\rho][\sigma]}\delta_{\gamma\alpha}\, \mathrm{dim}(H).\end{aligned} \tag{E.22}$$

If $F(\Lambda \times H)$ can be equipped with a co-algebra one can use the characters to obtain the fusion rules. We shall consider this in more detail in the next section.

E.3 FUSION RULES

In special case that Λ is a group, $F(\Lambda\times H)$ becomes a Hopf algebra with a co-multiplication defined by (E.8). The co-multiplication defines the tensor product of two representations and thus also the character of the tensor product. Since the characters of the irreducible representations are orthogonal one can compute the decomposition of the tensor product into irreducible representations by calculating the inner product of the characters of the irreducible representations with the character of the tensor product. In this way one obtains the fusion rules for the generalised transformation group algebra. We shall show that for Λ equal to the set of characters of an abelian group N the fusion rules of $F(\Lambda \times H)$ equal the fusion rules of $H \ltimes N$.
From equation (E.8) one find for the character $\chi_{a\times b}$ of the representation $a \otimes b$

$$\chi_{a\otimes b}(P_\lambda h) = \chi_a \otimes \chi_b(\Delta(P_\lambda h)) = \sum_{\kappa\in\Lambda} \chi_a(P_{\lambda\kappa^{-1}}h)\chi_b(P_\kappa k). \tag{E.23}$$

If we take the irreducible representations $a = \Pi_\alpha^{[\sigma]}$, $b = \Pi_\beta^{[\eta]}$ and $c = \Pi_\gamma^{[\rho]}$ we get from equations (E.20) and (E.21):

$$\begin{aligned}
\langle \chi_c, \chi_{a\otimes b}\rangle \quad &= \sum_{\lambda\in\Lambda} \int_H \chi_c(P_\lambda h)\chi^*_{a\otimes b}(P_\lambda h)dh \\
&= \sum_{\lambda\in\Lambda} \int_H \chi_c(P_\lambda h) \sum_{\kappa\in\Lambda} \chi^*_a(P_{\lambda\kappa^{-1}} h)\chi^*_b(P_\kappa k)dh \\
&= \sum_{\kappa\in\Lambda}\sum_{\lambda\in\Lambda} \int_H \sum_{\mu\in[\rho]} \delta_{\mu,\lambda}\delta_{h\triangleright\lambda,\lambda}\chi_\gamma(h_\lambda^{-1} h h_\lambda)\times \\
&\quad \sum_{\nu\in[\sigma]} \delta_{\nu,\lambda\kappa^{-1}}\delta_{h\triangleright(\lambda\kappa^{-1}),\lambda\kappa^{-1}}\chi^*_\alpha(h^{-1}_{\lambda\kappa^{-1}} h h_{\lambda\kappa^{-1}})\times \\
&\quad \sum_{\zeta\in[\eta]} \delta_{\zeta,\kappa}\delta_{h\triangleright\kappa,\kappa}\chi^*_\beta(h_\kappa^{-1} h h_\kappa)dh \\
&= \int_H \sum_{\mu\in[\rho]} \delta_{h\triangleright\mu,\mu}\chi_\gamma(h_\mu^{-1} h h_\mu)\times \\
&\quad \sum_{\nu\in[\sigma]} \delta_{\nu,\mu\zeta^{-1}}\delta_{h\triangleright\nu,\nu}\chi^*_\alpha(h_\nu^{-1} h h_\nu) \sum_{\zeta\in[\eta]} \delta_{h\triangleright\zeta,\zeta}\chi^*_\beta(h_\zeta^{-1} h h_\zeta)dh \\
&= \sum_{\mu\in[\rho]}\sum_{\nu\in[\sigma]}\sum_{\zeta\in[\eta]} \delta_{\mu,\nu\zeta} \int_H \delta_{h\triangleright\mu,\mu}\delta_{h\triangleright\nu,\nu}\delta_{h\triangleright\zeta,\zeta} \\
&\quad \chi_\gamma(h_\mu^{-1} h h_\mu)\chi^*_\alpha(h_\nu^{-1} h h_\nu)\chi^*_\beta(h_\zeta^{-1} h h_\zeta)dh.
\end{aligned} \tag{E.24}$$

Comparing this with equation (4.42), which only differs with an irrelevant constant factor, we conclude that this does indeed give the fusion rules of $H \ltimes N$.

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