Axiomatising Dynamic Logics for Anaphora

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Abstract

A task put on the agenda of philosophical logic by the dynamic turn in natural language analysis and in logic, the account for the dynamics of anaphoric context shifts in reasoning, is taken up along new lines. A standard way to treat reasoning with dynamic anaphora logics is by means of translation to first order predicate logic. This paper gives direct sequent axiomatisations of some well known dynamic anaphora logics. The calculi are shown to be sound and complete. The calculi throw new light on the relation between various formats for representing anaphoric links by means of sharing of dynamically bound variables. The sequent approach to the dynamics of anaphoric context takes the notion of anaphoric context change from premiss to conclusion as starting point of the analysis, and does not rely on an implicit translation to first order predicate logic.

Keywords: Context, dynamics, predicate logic, natural language semantics, axiomatisation, reasoning with anaphora

1 Pronominal Reference in Reasoning

The 'dynamic shift' in logic has led to an increased awareness for the process character of aspects of reasoning. One of the new topics put on the logical agenda by Kamp [8] and others is anaphoric processing: the way in which possible antecedents for pronominal reference are set up in discourse. As always, awareness of new facts gives occasion for new challenges. Instead of just giving an account of valid reasoning, why not include an account of the way in which premisses can set up an 'anaphoric context' for use by pronouns in the conclusion:

1.1
Every man is mortal. Suppose Socrates is a man. $\models$ Then *he* is mortal.

1.2
If a man owns a house, then he owns a garden. Suppose Socrates is a man who owns a house. $\models$ Then *he* owns a garden.

1.3
If a man owns a house, then he owns a garden. If a man owns a garden, then he sprinkles it. $\models$ If a man owns a house, then there is a thing that he sprinkles.

The task for a suitably dynamicized version of logic is to account for the validity of the above inferences, *and* for the pronoun-antecedent linkings in the discourses that make up the reasoning.

The most streamlined version of proposals for a logical representation language for anaphoric linking is Dynamic Predicate Logic or DPL (Groenendijk and Stokhof [7]), the first order version of the representation language proposed in Barwise [1]. DPL was intended as a rational reconstruction of Discourse Representation Theory or DRT [8], which was believed (erroneously, I think) to have a non-compositional semantics. Whatever its *raison d'être*, the dynamic variation on predicate logic was discovered more than once or twice, witness the existence of a German version, virtually identical to [7]: see Staudacher [11].

The dynamic logic of Barwise [1], Staudacher [11] and Groenendijk and Stokhof [7] essentially represents introduction of new antecedents by means of random assignment to a variable. The meaning of $\exists x$ becomes the relation between variable states s, r with the property that s and r differ at most in their x value:

$$s[\![\exists x]\!]r \text{ iff } s \sim_x r.$$

The language L of DPL is like that of predicate logic, but with the connective $\wedge$ replaced with a connective ; for sequential composition. Let a set of constants C and a set of variables V be given, and assume that c ranges over C and v over V. Also, assume a set of predicate constants P^i_j, with i indicating arity, and let P range over this set. We will omit the arity superscripts, and rely on context to indicate the arity of a predicate.

DEFINITION 1.4 (L)

$$\begin{array}{rcl} t & ::= & c \mid v \\ \phi & ::= & \top \mid \exists v; \phi \mid Pt_1 \cdots t_n; \phi \mid \neg(\phi_1); \phi_2. \end{array}$$

Note that we have built into the language that ; creates a flat list structure. Alternatively, we could allow $\phi; \psi; \chi$ to be ambiguous between $(\phi; \psi); \chi$ and $\phi; (\psi; \chi)$, but then we would have to build associativity of ; into the calculus below. Another alternative would have been to define an ambiguous language, so that $\phi; \psi; \chi$ would have several construction trees. This would also work, but it would complicate the structural induction proofs over the formulas of our language.

The formula $\top$ is the nil formula that goes with the flat list formula constructor ;. Every formula has the form $\phi_1; \ldots; \phi_n; \top$, with $n \geq 0$. The trailing $\top$ will prove its usefulness in the recursive definitions and the structural induction arguments. We will abbreviate $\phi_1; \ldots; \phi_n; \top$, with $n > 0$, as $\phi_1; \ldots; \phi_n$. Also, we will omit unnecessary parentheses, writing $\neg(Pt_1 \cdots t_n)$ as $\neg Pt_1 \cdots t_n$, etcetera, and we will use the following abbreviations:

$$\begin{array}{rcl} \bot & := & \neg\top \\ \phi_1 \to \phi_2 & := & \neg(\phi_1; \neg\phi_2) \\ \forall v \phi & := & \neg(\exists v; \neg\phi). \end{array}$$

The key feature of dynamic predicate logic is the ability of the existential quantifier to bind variables outside its proper scope. Assuming for a moment that ; creates structured lists, consider the DPL text $(\exists x; Px); (\exists y; Qy); Rxy$. Here the x and y of Rxy are bound, by $\exists x$ and $\exists y$ respectively, outside of the 'proper scope' of the quantifiers, as indicated by the bracketings. It follows that variables can be viewed as anaphoric elements linked to a preceding existential quantifier that introduces a referent. It is precisely because of the fact that $(\exists x; Px); Qx$ and $\exists x; (Px; Qx)$

are semantically equivalent that we can get away with considering ; as a flat list constructor.

The semantics of DPL is given as a relation between input and output assignments, given a first order model $\mathcal{M}$. We use $s \sim_v r$ in case s and r differ at most in the value they assign to v, and $s \sim_X r$ ($X \subseteq V$) in case s and r differ at most in the values they assign to the variables in X. We use sI_Xr if $\forall x \in X : s(x) = r(x)$. Note that sI_Xr iff $s \sim_{(V-X)} r$.

DEFINITION 1.5 (Term evaluation for L)

$$\begin{array}{rcl} [\![c]\!]_s^{\mathcal{M}} & = & c^{\mathcal{M}} \\ [\![v]\!]_s^{\mathcal{M}} & = & s(v). \end{array}$$

DEFINITION 1.6 (Semantics of L)

$$\begin{array}{lll} s[\![\top]\!]^{\mathcal{M}}r & \text{iff} & s = r \\ s[\![\exists v; \phi]\!]^{\mathcal{M}}r & \text{iff} & \text{there is an } s' \text{ with } (s \sim_v s' \text{ and } s'[\![\phi]\!]^{\mathcal{M}}r) \\ s[\![Pt_1 \cdots t_n; \phi]\!]^{\mathcal{M}}r & \text{iff} & \langle [\![t_1]\!]_s^{\mathcal{M}}, \ldots, [\![t_n]\!]_s^{\mathcal{M}} \rangle \in P^{\mathcal{M}} \text{ and } s'[\![\phi]\!]^{\mathcal{M}}r \\ s[\![\neg(\phi_1); \phi_2]\!]^{\mathcal{M}}r & \text{iff} & \text{there is no } s' \text{ with } s[\![\phi]\!]^{\mathcal{M}}s', \text{ and } s[\![\phi]\!]^{\mathcal{M}}r. \end{array}$$

Note that in this semantics for DPL, a repeated assignment to a single variable by means of a repeated use of the same existential quantifier-variable combination blocks off the individual introduced by the first use of the quantifier from further anaphoric reference. After $\exists x; Px; \exists x; Qx$, the variable x will refer to the individual introduced by $\exists x; Qx$, and the individual introduced by $\exists x; Px$ has become inaccessible.

A suitable consequence relation for this language is the following:

DEFINITION 1.7
$\phi \models \psi :\Longleftrightarrow$ for all $\mathcal{M}, s, t$: if $s[\![\phi]\!]^{\mathcal{M}}t$ then there is an r with $t[\![\psi]\!]^{\mathcal{M}}r$.

Note that this consequence relation allows carrying anaphoric links from premiss to conclusion. For example: from 'a man walks and he talks' it follows that 'he talks':

$$\exists x; Mx; Wx; Tx \models Tx.$$

A DPL formula ϕ succeeds in a state s for model $\mathcal{M}$ in case for some r, $s[\![\phi]\!]^{\mathcal{M}}r$. The weakest precondition for success of a DPL formula ϕ gives what can be called the truth condition for that formula; it can be described as $\langle\phi\rangle\top$, where $\langle\phi\rangle$ is a 'modal' operator interpreted with accessibility relation $[\![\phi]\!]$.

The following translation procedure in a language of classical predicate logic with DPL modalities constitutes the full precondition calculus for DPL:

$$\begin{array}{rcl} \langle\top\rangle\psi & \leftrightarrow & \psi \\ \langle\exists v; \phi\rangle\psi & \leftrightarrow & \exists v\langle\phi\rangle\psi \\ \langle Pt_1 \cdots t_n; \phi\rangle\psi & \leftrightarrow & Pt_1 \cdots t_n \wedge \langle\phi\rangle\psi \\ \langle\neg(\phi_1); \phi_2\rangle\psi & \leftrightarrow & \neg\langle\phi_1\rangle\top \wedge \langle\phi_2\rangle\psi. \end{array}$$

This reduction to classical predicate logic provides a kind of translation 'in the context of a given postcondition'. Without the postcondition context, the translation procedure would break down for the existential quantifier case, for $\exists v$ *per se* has no counterpart in classical predicate logic, of course. The translation is in the spirit of systems of precondition reasoning for imperative programs. See Van Eijck and De Vries [4] for further discussion of this connection. Such a calculus has its use for reasoning about DPL, but although it has been sold to the general public as an axiomatisation of DPL (in Van Eijck [3]), it is closer to a translation procedure than to a real axiomatisation.

2 Substitution in DPL

In the calculus we are about to present, we need some notions about freedom and bondage. First of all, we need the notion of being freely substitutable. For this, we first define a function *var* on the terms and formulas.

DEFINITION 2.1 ($var(t), var(\phi)$)

$$\begin{array}{lcl}
var(c) & := & \emptyset \\
var(v) & := & \{v\} \\
var(\top) & := & \emptyset \\
var(\exists v; \phi) & := & \{v\} \cup var(\phi) \\
var(Pt_1 \cdots t_n; \phi) & := & \bigcup_{1 \leq i \leq n} var(t_i) \cup var(\phi) \\
var(\neg\phi_1; \phi_2) & := & var(\phi_1) \cup var(\phi_2).
\end{array}$$

DEFINITION 2.2 (t is free for v in ϕ)
t is free for v in $\top$. t is free for v in $\exists v; \psi$. t is free for v in $\exists w; \psi$ $(w \neq v)$ if $w \notin var(t)$ and t is free for v in ψ. t is free for v in $Pt_1 \cdots t_n; \psi$ if t is free for v in ψ. t is free for v in $(\neg\psi_1); \psi_2$ if t is free for v in ψ_1 and t is free for v in ψ_2.

The set of variables which have a fixed occurrence in a DPL formula is given by a function $fix : L \to \mathcal{P}V$. The set of variables which are introduced in a formula is given by a function $intro : L \to \mathcal{P}V$, and the set of variables which have a classically bound occurrence in a formula is given by a function $cbnd : L \to \mathcal{P}V$.

DEFINITION 2.3 (fix, intro, cbnd)

- $fix(\top) := \emptyset, intro(\top) := \emptyset, cbnd(\top) := \emptyset$.
- $fix(\exists v; \phi) := fix(\phi) - \{v\}$,
  $intro(\exists v; \phi) := \{v\} \cup intro(\phi)$,
  $cbnd(\exists v; \phi) := cbnd(\phi)$.
- $fix(Pt_1 \cdots t_n; \phi) := var(Pt_1 \cdots t_n) \cup fix(\phi)$,
  $intro(Pt_1 \cdots t_n; \phi) := intro(\phi)$,
  $cbnd(Pt_1 \cdots t_n; \phi) := cbnd(\phi)$.
- $fix(\neg(\phi_1); \phi_2) := fix(\phi_1) \cup fix(\phi_2)$,
  $intro(\neg(\phi_1); \phi_2) := intro(\phi_2)$,
  $cbnd(\neg(\phi_1); \phi_2) := intro(\phi_1) \cup cbnd(\phi_1) \cup cbnd(\phi_2)$.

Some examples may clarify this. Let $\phi := \exists v; \exists w; Ruvw$. Then $intro(\phi) = \{v, w\}$, $fix(\phi) = \{u\}$, $cbnd(\phi) = \emptyset$. Variables introduced within the scope of negation become classically bound. Let $\phi := \neg(\exists v; \exists w; Ruvw)$. Then $intro(\phi) = \emptyset$, $fix(\phi) = \{u\}$,

$cbnd(\phi) = \{v, w\}$. A variable can have fixed, bound and introduced occurrences in an expression. Let $\phi := Px; \exists x; \neg Px; \neg(\exists x; Qx)$. Then $intro(\phi) = \{x\}$, $fix(\phi) = \{x\}$, $cbnd(\phi) = \{x\}$.

We call an occurrence of v in ϕ *free* if it is not introduced and not classically bound. Thus, the leftmost occurrence of x in $Px; \exists x; \neg Px; \neg(\exists x; Qx)$ is free, the other occurrences are not. We use $[t/v]\phi$ for the result of substituting t for all free occurrences of v in ϕ:

DEFINITION 2.4 ($[t/v]\phi$)

$$\begin{array}{lcl}
[t/v]c & := & c \\
[t/v]v & := & t \\
[t/v]w & := & w \\
[t/v]\top & := & \top \\
[t/v](\exists v; \phi) & := & \exists v; \phi \\
[t/v](\exists w; \phi) & := & \exists w; [t/v]\phi \\
[t/v](Pt_1 \cdots t_n; \phi) & := & P[t/v]t_1 \cdots [t/v]t_n; [t/v]\phi \\
[t/v](\neg(\phi_1); \phi_2) & := & \neg([t/v]\phi_1); [t/v]\phi_2.
\end{array}$$

Note that this substitution takes the dynamic binding force of $\exists v$ over the text that follows into account (cf. the clause for $[t/v](\exists v; \phi)$, where the occurrence of $\exists v$ blocks off the ϕ that follows). Visser [14] calls this substitution 'left' substitution.

We use $s[v := a]$ for the function s' defined by $s'(w) = s(w)$ if $w \neq v$, $s'(w) = a$ if $w = v$.

LEMMA 2.5 (Term Substitution)

$$[\![[t_1/v]t_2]\!]_s^{\mathcal{M}} = [\![t_2]\!]_{s[v:=[\![t_1]\!]_s^{\mathcal{M}}]}^{\mathcal{M}}.$$

PROOF. In case t_2 is not a variable, $[t_1/v]t_2$ equals t_2, and the lemma trivially holds.

If $t_2 = w \neq v$, $[t_1/v]t_2$ equals t_2, and the lemma trivially holds.

If $t_2 = v$, $[t_1/v]t_2$ equals t_1, and the lemma holds because $[\![t_1]\!]_s^{\mathcal{M}} = s[v := [\![t_1]\!]_s^{\mathcal{M}}](v)$. ∎

The following formula substitution lemma makes clear that substitution in dynamic predicate logic behaves differently from substitution in standard predicate logic. When substituting into ϕ, we have to take the context of ϕ (in the form of the set of variables given by $intro(\phi)$) into account. For further elaboration of this theme we refer to Visser [14].

LEMMA 2.6 (Formula Substitution)
If t is free for v in ϕ and $v \in intro(\phi)$ then

$$s[\![[t/v]\phi]\!]^{\mathcal{M}} r \Leftrightarrow s[v := [\![t]\!]_s^{\mathcal{M}}][\![\phi]\!]^{\mathcal{M}} r.$$

If t is free for v in ϕ and $v \notin intro(\phi)$ then

$$s[\![[t/v]\phi]\!]^{\mathcal{M}} r \Leftrightarrow s[v := [\![t]\!]_s^{\mathcal{M}}][\![\phi]\!]^{\mathcal{M}} r[v := [\![t]\!]_s^{\mathcal{M}}].$$

PROOF. Induction on the structure of ϕ.

$\phi = \top$. OK.

$\phi = \exists v;\psi$. Then $v \in \mathit{intro}(\phi)$, and

$$\begin{array}{ll} & s[\![[t/v]\exists v;\psi]\!]^{\mathcal{M}} r \\ \iff & s[\![\exists v;\psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_v s' \text{ and } s'[\![\psi]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![\exists v;\psi]\!]^{\mathcal{M}} r. \end{array}$$

$\phi = \exists w;\psi$, with $w \neq v$. Suppose $v \in \mathit{intro}(\phi)$. Then $v \in \mathit{intro}(\psi)$, and

$$\begin{array}{ll} & s[\![[t/v]\exists w;\psi]\!]^{\mathcal{M}} r \\ \iff & s[\![\exists w;[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_w s' \text{ and } s'[\![[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_w s' \text{ and } s'[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![\exists w;\psi]\!]^{\mathcal{M}} r \end{array}$$

$\phi = \exists w;\psi$, with $w \neq v$. Suppose $v \notin \mathit{intro}(\phi)$. Then $v \notin \mathit{intro}(\psi)$, and

$$\begin{array}{ll} & s[\![[t/v]\exists w;\psi]\!]^{\mathcal{M}} r \\ \iff & s[\![\exists w;[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_w s' \text{ and } s'[\![[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s \sim_w s' \text{ and } s'[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi]\!]^{\mathcal{M}} r[v := [\![t]\!]^{\mathcal{M}}_s] \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![\exists w;\psi]\!]^{\mathcal{M}} r[v := [\![t]\!]^{\mathcal{M}}_s] \end{array}$$

$\phi = Pt_1 \cdots t_n;\psi$. Suppose $v \in \mathit{intro}(\phi)$. Then $v \in \mathit{intro}(\psi)$, and

$$\begin{array}{ll} & s[\![[t/v](Pt_1 \cdots t_n;\psi)]\!]^{\mathcal{M}} r \\ \iff & s[\![[t/v]Pt_1 \cdots t_n;[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & \langle [\![[t/v]t_1]\!]^{\mathcal{M}}_s, \ldots, [\![[t/v]t_n]\!]^{\mathcal{M}}_s \rangle \in P^{\mathcal{M}} \text{ and } s[\![[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & \langle [\![t_1]\!]^{\mathcal{M}}_{s[v:=[\![t]\!]^{\mathcal{M}}_s]}, \ldots, [\![t_n]\!]^{\mathcal{M}}_{s[v:=[\![t]\!]^{\mathcal{M}}_s]} \rangle \in P^{\mathcal{M}} \text{ and } s[\![[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![Pt_1 \cdots t_n]\!]^{\mathcal{M}} s[v := [\![t]\!]^{\mathcal{M}}_s] \text{ and } s[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![Pt_1 \cdots t_n;\psi]\!]^{\mathcal{M}} r \end{array}$$

$\phi = Pt_1 \cdots t_n;\psi$. Suppose $v \notin \mathit{intro}(\phi)$. Then $v \notin \mathit{intro}(\psi)$, and

$$\begin{array}{ll} & s[\![[t/v](Pt_1 \cdots t_n;\psi)]\!]^{\mathcal{M}} r \\ \iff & s[\![[t/v]Pt_1 \cdots t_n;[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & \langle [\![[t/v]t_1]\!]^{\mathcal{M}}_s, \ldots, [\![[t/v]t_n]\!]^{\mathcal{M}}_s \rangle \in P^{\mathcal{M}} \text{ and } s[\![[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & \langle [\![t_1]\!]^{\mathcal{M}}_{s[v:=[\![t]\!]^{\mathcal{M}}_s]}, \ldots, [\![t_n]\!]^{\mathcal{M}}_{s[v:=[\![t]\!]^{\mathcal{M}}_s]} \rangle \in P^{\mathcal{M}} \text{ and } s[\![[t/v]\psi]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![Pt_1 \cdots t_n]\!]^{\mathcal{M}} s[v := [\![t]\!]^{\mathcal{M}}_s] \text{ and } s[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi]\!]^{\mathcal{M}} r[v := [\![t]\!]^{\mathcal{M}}_s] \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![Pt_1 \cdots t_n;\psi]\!]^{\mathcal{M}} r[v := [\![t]\!]^{\mathcal{M}}_s] \end{array}$$

$\phi = \neg(\psi_1);\psi_2$. Suppose $v \in \mathit{intro}(\phi)$. Then $v \in \mathit{intro}(\psi_2)$, and

$$\begin{array}{ll} & s[\![[v/t]\neg(\psi_1);\psi_2]\!]^{\mathcal{M}} r \\ \iff & s[\![\neg([v/t]\psi_1);[v/t]\psi_2]\!]^{\mathcal{M}} r \\ \iff & \text{there is no } s' \text{ with } s[\![[v/t]\psi_1]\!]^{\mathcal{M}} s', \text{ and } s[\![[v/t]\psi_2]\!]^{\mathcal{M}} r \\ \iff & \text{there is no } s' \text{ with } s[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi]\!]^{\mathcal{M}} s' \text{ and } s[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi_2]\!]^{\mathcal{M}} r \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![\neg(\psi_1);\psi_2]\!]^{\mathcal{M}} r. \end{array}$$

$\phi = \neg(\psi_1);\psi_2$. Suppose $v \notin \mathit{intro}(\phi)$. Then $v \notin \mathit{intro}(\psi_2)$, and

$$\begin{array}{ll} & s[\![[v/t]\neg(\psi_1);\psi_2]\!]^{\mathcal{M}}r \\ \iff & s[\![\neg([v/t]\psi_1);[v/t]\psi_2]\!]^{\mathcal{M}}r \\ \iff & \text{there is no } s' \text{ with } s[\![[v/t]\psi_1]\!]^{\mathcal{M}}s',\text{ and } s[\![[v/t]\psi_2]\!]^{\mathcal{M}}r \\ \iff & \text{there is no } s' \text{ with } s[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi]\!]^{\mathcal{M}}s' \text{ and } s[v := [\![t]\!]^{\mathcal{M}}_s][\![\psi_2]\!]^{\mathcal{M}}r[v := [\![t]\!]^{\mathcal{M}}_s] \\ \iff & s[v := [\![t]\!]^{\mathcal{M}}_s][\![\neg(\psi_1);\psi_2]\!]^{\mathcal{M}}r[v := [\![t]\!]^{\mathcal{M}}_s]. \end{array}$$

■

The following lemmas use the functions *intro* and *fix* (and indeed motivate their definition).

LEMMA 2.7
$s[\![\phi]\!]^{\mathcal{M}}t$ implies $s \sim_{\mathit{intro}(\phi)} t$.

PROOF. Induction on the structure of ϕ. ■

LEMMA 2.8
If $s[\![\phi]\!]^{\mathcal{M}}s'$ and $sI_X r$ for $X \supseteq \mathit{fix}(\phi)$ then there is an r' with $r[\![\phi]\!]^{\mathcal{M}}r'$.

PROOF. Induction on the structure of ϕ.

$\top$: trivial.

$\exists v;\psi$. Then $s[\![\exists v;\psi]\!]^{\mathcal{M}}s'$, and there is an s'' with $s \sim_v s''$ and $s''[\![\psi]\!]^{\mathcal{M}}s'$. Take $r'' = r[v := s''(v)]$. Then, by the induction hypothesis, there is an r' with $r''[\![\psi]\!]^{\mathcal{M}}r'$, and therefore $r[\![\exists v;\psi]\!]^{\mathcal{M}}r'$.

$Pt_1\cdots t_n;\psi$. Then we have $s[\![Pt_1\cdots t_n;\psi]\!]^{\mathcal{M}}s'$ iff $s[\![Pt_1\cdots t_n]\!]^{\mathcal{M}}s$ and $s[\![\psi]\!]^{\mathcal{M}}s'$ iff $r[\![Pt_1\cdots t_n]\!]^{\mathcal{M}}r$ (by the fact that $sI_{\mathit{var}(Pt_1\cdots t_n)}r$) and $r[\![\psi]\!]^{\mathcal{M}}r'$ for some r' (by i.h.) iff $r[\![Pt_1\cdots t_n;\psi]\!]^{\mathcal{M}}r'$.

$\neg(\phi_1);\phi_2$. Then $s[\![\neg(\psi_1);\psi_2]\!]^{\mathcal{M}}s'$ iff $s[\![\neg\psi_1]\!]^{\mathcal{M}}s$ and $s[\![\psi_2]\!]^{\mathcal{M}}s'$. By the induction hypothesis $r[\![\neg\psi_1]\!]^{\mathcal{M}}r$. and by a second application of the induction hypothesis there is an r' with $r[\![\psi_2]\!]^{\mathcal{M}}r'$. Thus, there is an r' with $r[\![\neg(\psi_1);\psi_2]\!]^{\mathcal{M}}r'$. ■

3 A Calculus for DPL

In this section, we will give a set of sequent deduction rules for DPL. We will use $\phi \Longrightarrow \psi$, where $\Longrightarrow$ is the sequent separator. Note that $\phi \Longrightarrow \bot$ expresses that ϕ is inconsistent. The calculus defines a relation $\Longrightarrow \subseteq L^2$ by means of: $\phi \Longrightarrow \psi$ iff $\phi \Longrightarrow \psi$ is at the root of a finite tree with sequents at its nodes, such that the sequents at a leaf node are axioms of the calculus, and the sequents at the internal nodes follow by means of a rule of the calculus from the sequent(s) at the daughter node(s) of that internal node.

In the calculus, we use C, with and without subscripts, as a variable over contexts, where a context is a formula or the empty list ϵ. Substitution and evaluation are extended to contexts in the obvious way. If C is a context and ϕ a formula, then we use $C\phi$ for the formula given by: $C\phi := \phi$ if $C = \epsilon$, $C\phi := \psi;\phi$ if $C = \psi$. Similarly for ϕC, and for $C_1\phi C_2$.

It is convenient to extend the definition of substitution to sequents.

FIG. 1. The Calculus for DPL

test axiom	$\dfrac{}{T \Longrightarrow T}$
transitivity	$\dfrac{\phi \Longrightarrow \psi \qquad \psi \Longrightarrow \chi}{\phi \Longrightarrow \chi}\ \mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$
test swap	$\dfrac{C_1 T_1; T_2 C_2 \Longrightarrow \phi}{C_1 T_2; T_1 C_2 \Longrightarrow \phi}$
quantifier move	$\dfrac{C_1 T; \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists v; T C_2 \Longrightarrow \phi}\ v \notin \mathit{fix}(T) \qquad \dfrac{C_1 \exists v; T C_2 \Longrightarrow \phi}{C_1 T; \exists v\ C_2 \Longrightarrow \phi}\ v \notin \mathit{fix}(T)$
quantifier intro	$\dfrac{\phi \Longrightarrow [t/v]\psi}{\phi \Longrightarrow \exists v; \psi}$
var refreshment	$\dfrac{C_1 \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists w [w/v] C_2 \Longrightarrow \phi}\ w \notin \mathit{intro}(C_1) \cup \mathit{fix}(C_1)$
sequencing	$\dfrac{\psi \Longrightarrow \chi}{\phi; \psi \Longrightarrow \chi} \qquad \dfrac{\phi \Longrightarrow \psi \qquad \phi \Longrightarrow \chi}{\phi \Longrightarrow \psi; \chi}\ \mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$
negation	$\dfrac{\phi \Longrightarrow \psi}{\phi; \neg\psi \Longrightarrow \bot} \qquad \dfrac{\phi; \psi \Longrightarrow \bot}{\phi \Longrightarrow \neg\psi}$
double negation	$\dfrac{\phi \Longrightarrow \neg\neg\psi}{\phi \Longrightarrow \psi} \qquad \dfrac{\phi; \neg\neg\psi \Longrightarrow \bot}{\phi; \psi \Longrightarrow \bot}$

DEFINITION 3.1 ($[t/v]C \Longrightarrow \phi$)
Induction on the structure of C

$$[t/v]\epsilon \Longrightarrow \phi \quad := \quad \epsilon \Longrightarrow [t/v]\phi$$

$$[t/v]\psi \Longrightarrow \phi \quad := \quad \begin{cases} ([t/v]\psi) \Longrightarrow \phi & \text{if } v \in \mathit{intro}(\psi) \\ ([t/v]\psi) \Longrightarrow ([t/v]\phi) & \text{otherwise.} \end{cases}$$

Substitution for sequents carries in its wake a notion of being free for a variable in a sequent:

DEFINITION 3.2 (t is free for v in $C \Longrightarrow \psi$)

1. t is free for v in $\epsilon \Longrightarrow \psi$ if t is free for v in ψ.
2. t is free for v in $\phi \Longrightarrow \psi$ if t is free for v in ϕ, and either $v \in \mathit{intro}(\phi)$ or t is free for v in ψ.

When a rule mentions a substitution $[t/v]\phi$ in the consequent of a sequent then the standard assumption is made hat t is free for v in ϕ. Note that in the rules below, $\Longrightarrow$ binds more strongly than $[t/v]$.When a rule mentions a substitution $C_1[t/v]C_2 \Longrightarrow \phi$ then it is assumed that t is free for v in $C_2 \Longrightarrow \phi$.

In the rules below we will use T as an abbreviation of formulas ϕ with $intro(\phi) = \emptyset$ (T for *Test* formula).

Structural Rules

Test Axiom

$$\frac{}{T \Longrightarrow T}$$

Soundness of Test Axiom

If $s[\![T]\!]^{\mathcal{M}}t$ then $s = t$ (because T is a test) and thus there is an r with $t[\![T]\!]^{\mathcal{M}}r$, namely $r = s$. Thus, $T \models T$.

Transitivity Rule

$$\frac{\phi \Longrightarrow \psi \qquad \psi \Longrightarrow \chi}{\phi \Longrightarrow \chi}\ intro(\psi) \cap fix(\chi) = \emptyset$$

In case the side condition is not fulfilled, this can always be remedied by one or more applications of $\exists$ right (see below). Note that in the case where the 'cut' formula is a test the side condition of the rule is always fulfilled, and the rule assumes the familiar format:

$$\frac{\phi \Longrightarrow T \qquad T \Longrightarrow \chi}{\phi \Longrightarrow \chi}$$

Here is an example application:

$$\frac{Rxy \Longrightarrow Ryx \qquad Ryx \Longrightarrow \exists z; Ryz}{Rxy \Longrightarrow \exists z; Ryz}\ \text{trans}$$

Soundness of Transitivity Rule

Suppose $\phi \models \psi$ and $\psi \models \chi$. Assume $s[\![\phi]\!]^{\mathcal{M}}t$. Then by $\phi \models \psi$, there is an r with $t[\![\psi]\!]^{\mathcal{M}}r$. By $\psi \models \chi$, there is a u with $r[\![\chi]\!]^{\mathcal{M}}u$. Then by Lemma 2.7, $t \sim_{intro(\psi)} r$. By the fact that $intro(\psi) \cap fix(\chi) = \emptyset$, $tI_{fix(\chi)}r$. By $tI_{fix(\chi)}r$, $r[\![\chi]\!]^{\mathcal{M}}u$ and Lemma 2.8 there is an u' with $t[\![\chi]\!]^{\mathcal{M}}u'$. Thus, $\phi \models \chi$.

Weakening Rules

Due to the format where ; serves as the concatenation operator for formulas, the rule for ; left does double duty as an antecedent weakening rule. See below. Succedent weakening would be the step from $\phi \Longrightarrow \psi$ to $\phi \Longrightarrow \neg(\neg\psi; \neg\chi)$. This is taken care of by the negation rules. Again: see below.

Test Swap Rule

$$\frac{C_1 T_1; T_2 C_2 \Longrightarrow \phi}{C_1 T_2; T_1 C_2 \Longrightarrow \phi}$$

Soundness of Test Swap Rules

Follows from the fact that $s[\![T_1; T_2]\!]^{\mathcal{M}} t$ iff $s[\![T_2; T_1]\!]^{\mathcal{M}} t$.

Quantifier Movement Rules

$$\frac{C_1 T; \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists v; T C_2 \Longrightarrow \phi}\ v \notin \mathit{fix}(T) \qquad \frac{C_1 \exists v; T C_2 \Longrightarrow \phi}{C_1 T; \exists v\ C_2 \Longrightarrow \phi}\ v \notin \mathit{fix}(T)$$

These rules allow us to pull $\exists v$ through a test T. To avoid accidental capture of variables, the condition has to be fulfilled that $\exists v$ does not bind anything in T, in other words, that $v \notin \mathit{fix}(T)$.

Soundness of Quantifier Movement Rules

The crucial observation is that for a test T with $v \notin \mathit{fix}(T)$ we have that for all models $\mathcal{M}$, $[\![\exists v; T]\!]^{\mathcal{M}} = [\![T; \exists v]\!]^{\mathcal{M}}$.

$\exists$ Left

This is a special case of ; left. See below.

Quantifier Intro

$$\frac{\phi \Longrightarrow [t/v]\psi}{\phi \Longrightarrow \exists v; \psi}$$

Note that it is assumed that t is free for v in ψ. This assumption is familiar from the Gentzen format of $\exists$-right in standard predicate logic. Here is an example application:

$$\frac{Rxx \Longrightarrow Rxx}{Rxx \Longrightarrow \exists y; Rxy}$$

Rxx equals $[x/y]Rxy$, so this is indeed a correct application of the rule.

Soundness of Quantifier Intro

Assume $s[\![\phi]\!]^{\mathcal{M}} s_1$. Then by the soundness of the premiss there is an s_2 with

$$s_1[\![[t/v]\psi]\!]^{\mathcal{M}} s_2.$$

Assume $v \in \mathit{intro}(\psi)$. Then by the substitution lemma,

$$s_1[v := [\![t]\!]^{\mathcal{M}}_{s_1}][\![\psi]\!]^{\mathcal{M}} s_2,$$

and thus there is an r with $s_1 \sim_v r$ and $r[\![\psi]\!]^{\mathcal{M}} s_2$. Assume $v \notin \mathit{intro}(\psi)$. Then by the substitution lemma,

$$s_1[v := [\![t]\!]^{\mathcal{M}}_{s_1}][\![\psi]\!]^{\mathcal{M}} s_2[v := [\![t]\!]^{\mathcal{M}}_{s_1}],$$

and thus there is an r with $s_1 \sim_v r$ and $r[\![\psi]\!]^{\mathcal{M}} s_2[v := [\![t]\!]^{\mathcal{M}}_{s_1}]$. In both cases there is a t with $s_1[\![\exists v;\psi]\!]^{\mathcal{M}} t$. This establishes $\phi \models \exists v;\psi$.

Variable Refreshment Rule

$$\frac{C_1 \exists v\ C_2 \Longrightarrow \phi}{C_1 \exists w [w/v] C_2 \Longrightarrow \phi}\ w \notin \mathit{intro}(C_1) \cup \mathit{fix}(C_1)$$

Variable refreshment allows the liberation of a captured variable, e.g., of the first two occurrences of x in $\exists x; Px; \exists x; Qx$, by means of replacement by a variable that does not occur as an introduced or active variable in the left context in the given sequent:

$$\frac{\exists x; Px; \exists x; Qx \Longrightarrow Qx}{\exists y; Py; \exists x; Qx \Longrightarrow Qx}$$

It is also possible to change the other occurrences of x in the same example. The following is also a correct application of the rule:

$$\frac{\exists x; Px; \exists x; Qx \Longrightarrow Qx}{\exists x; Px; \exists y; Qy \Longrightarrow Qy}$$

Note that the rule can also be used to recycle a variable:

$$\frac{\exists y; Py; \exists x; Qx \Longrightarrow Qx}{\exists x; Px; \exists x; Qx \Longrightarrow Qx}$$

This application is also correct, for

$$(\exists x; Px; \exists x; Qx \Longrightarrow Qx) = (\exists x; [x/y]Py; \exists x; Qx \Longrightarrow Qx).$$

Soundness of Variable Refreshment

Assume $s[\![C_1 \exists w [w/v] C_2]\!]^{\mathcal{M}} r$. Then there are s_1, s_2 with

$$s[\![C_1]\!]^{\mathcal{M}} s_1, s_1 \sim_w s_2, s_2[\![[w/v]C_2]\!]^{\mathcal{M}} r.$$

Let $s' := s[w := s_2(w)]$ and $s'_1 := s_1[w := s_2(w)]$. Because $w \notin \mathit{intro}(C_1)$ and $w \notin \mathit{fix}(C_1)$, we get by Lemmas 2.7 and 2.8 that $s'[\![C_1]\!]^{\mathcal{M}} s'_1$. From $s_1 \sim_w s_2$ we get that $s'_1 = s_2$, and therefore $s'_1 \sim_v s_2[v := s_2(w)]$. Suppose $v \in \mathit{intro}(C_2)$. Then by Lemma 2.6, $s_2[v := s_2(w)][\![C_2]\!]^{\mathcal{M}} r$. Therefore $s'[\![C_1 \exists v\ C_1]\!]^{\mathcal{M}} r$. By the soundness of the premiss it follows from $s'[\![C_1 \exists v C_1]\!]^{\mathcal{M}} r$ that there is a t with $r[\![\phi]\!]^{\mathcal{M}} t$. This establishes $C_1 \exists w; [w/v] C_2 \models \phi$. Suppose on the other hand that $v \notin \mathit{intro}(C_2)$. Then the substitution lemma yields that $s_2[v := s_2(w)][\![C_2]\!]^{\mathcal{M}} r[v := s_2(w)]$. Therefore $s'[\![C_1 \exists v\ C_1]\!]^{\mathcal{M}} r[v := s_2(w)]$. By the soundness of the premiss it follows from $s'[\![C_1 \exists v C_1]\!]^{\mathcal{M}} r[v := s_2(w)]$ that there is a t with $r[v := s_2(w)][\![\phi]\!]^{\mathcal{M}} t$. By another application of the substitution lemma we get from this that there is a t' with $r[\![[w/v]\phi]\!]^{\mathcal{M}} t'$. This establishes $C_1 \exists w; [w/v] C_2 \models [w/v]\phi$.

; Left and Right

$$\frac{\psi \Longrightarrow \chi}{\phi; \psi \Longrightarrow \chi} \qquad\qquad \frac{\phi \Longrightarrow \psi \qquad \phi \Longrightarrow \chi}{\phi \Longrightarrow \psi; \chi}\ \mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$$

The first of these does double duty as a left weakening rule. Antecedent weakening is always extension on the lefthand side. This is because extension on the righthand-side might introduce new variable bindings that extend to the consequent. Weakening with a test is valid anywhere in the antecedent; the swap rules account for that.

An example application of the rule for ; right is:

$$\frac{Rxx \Longrightarrow \exists y; Ryx \qquad\qquad Rxx \Longrightarrow \exists z; Rxz}{Rxx \Longrightarrow \exists y; Ryx; \exists z; Rxz} \ ;\ \text{right}$$

In case the condition on the rule for ; right is not satisfied, e.g. for the two sequents $\neg Px; \exists x; Px \Longrightarrow \exists x; \neg Px$ and $\neg Px; \exists x; Px \Longrightarrow Px$, this can always be remedied by one or more applications of $\exists$ Right to the second premiss.

Soundness of ; Left

Suppose $s[\![\phi;\psi]\!]^{\mathcal{M}}t$. Then there is an s' with $s[\![\phi]\!]^{\mathcal{M}}s'$ and $s'[\![\psi]\!]^{\mathcal{M}}t$. From $s'[\![\psi]\!]^{\mathcal{M}}t$ and the soundness of the premiss it follows that there is an r with $t[\![\chi]\!]^{\mathcal{M}}r$. This proves that $\phi;\psi \models \chi$.

Soundness of ; Right

Assume $s[\![\phi]\!]^{\mathcal{M}}t$. By the soundness of the first premiss, there is an r with $t[\![\psi]\!]^{\mathcal{M}}r$. By the soundness of the second premiss, there is an r' with $t[\![\chi]\!]^{\mathcal{M}}r'$.

From $t[\![\psi]\!]^{\mathcal{M}}r$ we get by Lemma 2.7 that $t \sim_{\mathit{intro}(\psi)} r$ and from the fact that $\mathit{intro}(\psi) \cap \mathit{fix}(\chi) = \emptyset$ we may conclude, by Lemma 2.8, that there is an r'' with $r[\![\chi]\!]^{\mathcal{M}}r''$, and we have established that $\phi \models \psi;\chi$.

$\neg$ Left and Right

$$\frac{\phi \Longrightarrow \psi}{\phi; \neg\psi \Longrightarrow \bot} \qquad\qquad \frac{\phi;\psi \Longrightarrow \bot}{\phi \Longrightarrow \neg\psi}$$

Soundness of $\neg$ Left

Assume $s[\![\phi;\neg\psi]\!]^{\mathcal{M}}t$. Then $s[\![\phi]\!]^{\mathcal{M}}t$ and there is no r with $t[\![\psi]\!]^{\mathcal{M}}r$. Contradiction with the soundness of the premiss. This establishes $\phi;\neg\psi \models \bot$.

Soundness of $\neg$ Right

Assume $s[\![\phi]\!]^{\mathcal{M}}t$. Then by the soundness of the premiss, there is no r with $t[\![\psi]\!]^{\mathcal{M}}r$. This establishes $\phi \models \neg\psi$.

Double Negation Rules

$$\frac{\phi \Longrightarrow \neg\neg\psi}{\phi \Longrightarrow \psi} \qquad\qquad \frac{\phi; \neg\neg\psi \Longrightarrow \bot}{\phi;\psi \Longrightarrow \bot}$$

Soundness of Double Negation Rules

For Double Negation Left, assume $s[\![\phi]\!]^{\mathcal{M}}t$. Then by the soundness of the premiss, there is no r with $t[\![\neg\psi]\!]^{\mathcal{M}}r$. By the fact that $\neg\psi$ is a test, this means that it is not the case that $t[\![\neg\psi]\!]^{\mathcal{M}}t$. But then there is an r with $t[\![\psi]\!]^{\mathcal{M}}r$. This establishes $\phi \models \psi$. The soundness of Double Negation Right is established similarly.

This completes the presentation of the calculus. As we have checked the soundness of every rule as we went along, we have:

THEOREM 3.3
The Calculus for DPL is sound.

PROOF. Induction on the structure of a proof tree for $\phi \Longrightarrow \psi$. ∎

4 Derivable Rules for DPL Reasoning

PROPOSITION 4.1 (Contradiction Rule)
The following rule is derivable:

$$\frac{\phi;\neg\psi \Longrightarrow \neg\chi \qquad \phi;\neg\psi \Longrightarrow \chi}{\phi \Longrightarrow \psi}$$

PROOF. Here is a derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi;\neg\psi \Longrightarrow \neg\chi \qquad \phi;\neg\psi \Longrightarrow \chi}{\phi;\neg\psi \Longrightarrow \neg\chi;\chi}\;;\text{r} \qquad \dfrac{\dfrac{\dfrac{}{\neg\chi \Longrightarrow \neg\chi}\;\text{test axiom}}{\neg\chi;\neg\neg\chi \Longrightarrow \bot}\;\neg\text{l}}{\neg\chi;\chi \Longrightarrow \bot}\;\text{dn}}{\phi;\neg\psi \Longrightarrow \bot}\;\text{tr}}{\phi \Longrightarrow \neg\neg\psi}\;\neg\text{r}}{\phi \Longrightarrow \psi}\;\text{dn}$$

∎

PROPOSITION 4.2 (Cases Rule)
The following rule is derivable:

$$\frac{\phi;\neg\psi \Longrightarrow \chi \qquad \phi;\neg\neg\psi \Longrightarrow \chi}{\phi \Longrightarrow \chi}$$

PROOF. Here is a derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi;\neg\psi \Longrightarrow \chi}{\phi;\neg\psi;\neg\chi \Longrightarrow \bot}\;\neg\text{l}}{\phi;\neg\chi;\neg\psi \Longrightarrow \bot}\;\text{swap}}{\phi;\neg\chi \Longrightarrow \neg\neg\psi}\;\neg\text{r} \qquad \dfrac{\dfrac{\dfrac{\phi;\neg\neg\psi \Longrightarrow \chi}{\phi;\neg\neg\psi;\neg\chi \Longrightarrow \bot}\;\neg\text{l}}{\phi;\neg\chi;\neg\neg\psi \Longrightarrow \bot}\;\text{swap}}{\phi;\neg\chi \Longrightarrow \neg\neg\neg\psi}\;\neg\text{r}}{\phi \Longrightarrow \chi}\;\text{contrad}$$

∎

PROPOSITION 4.3 (Inconsistency Rule)
The following rule is derivable:

$$\frac{\phi \Longrightarrow \bot}{\phi \Longrightarrow \psi}$$

PROOF. Here is a derivation:

$$\dfrac{\phi \Longrightarrow \bot \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\bot \Longrightarrow \bot}\,\text{test}}{\neg\psi; \bot \Longrightarrow \bot}\,;\text{l}}{\bot; \neg\psi \Longrightarrow \bot}\,\text{swap}}{\bot \Longrightarrow \neg\neg\psi}\,\neg\text{r}}{\bot \Longrightarrow \psi}\,\text{dn}}{\phi \Longrightarrow \psi}\,\text{tr}$$

∎

PROPOSITION 4.4 (Modus Ponens)
The following rule is derivable:

$$\dfrac{\phi \Longrightarrow \neg\neg(\psi) \to \chi \qquad \phi \Longrightarrow \psi}{\phi \Longrightarrow \chi}$$

PROOF. Here is a derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \Longrightarrow \neg(\neg\neg(\psi); \neg\chi)}{\phi; \neg\neg(\neg\neg(\psi); \neg\chi) \Longrightarrow \bot}\,\neg\text{l}}{\phi; \neg\neg\psi; \neg\chi \Longrightarrow \bot}\,\text{dn}}{\phi; \neg\neg\psi \Longrightarrow \neg\neg\chi}\,\neg\text{r}}{\phi; \neg\neg\psi \Longrightarrow \chi}\,\text{dn} \qquad \dfrac{\dfrac{\phi \Longrightarrow \psi}{\phi; \neg\psi \Longrightarrow \bot}\,\neg\text{l}}{\phi; \neg\psi \Longrightarrow \chi}\,\text{incons}}{\phi \Longrightarrow \chi}\,\text{cases}$$

∎

We will need these derived rules in the completeness proof below.

5 Completeness of the Calculus

To establish the completeness of the calculus, assume that $\phi \not\Longrightarrow \psi$. We will construct a countermodel by a slight modification of the standard Henkin construction for the completeness of classical predicate logic.

We will define a relation $\phi \vdash_\Gamma \psi$ in such a way that the extra premisses from Γ do not extend the 'anaphoric context': the context change potential of the premisses from Γ is blocked off by means of double negation signs. This is a key element in the canonical model construction below.

A peculiarity of the completeness argument is that we have to construct a valuation in the canonical model. This is because the completeness theorem is a statement about open formulas, not sentences. To refute $\phi \models \psi$ we use a canonical model $\mathcal{M}_\Gamma$ and a canonical valuation i such that there is a valuation s with $s[\![\phi]\!]^{\mathcal{M}_\Gamma} i$, and for no r the relation $i[\![\psi]\!]^{\mathcal{M}_\Gamma} r$ holds. By taking for our objects in the canonical model all variables of the language, plus enough witnessing constants, we can get away with taking the canonical valuation i to be the identity valuation: the valuation that maps every variable to itself.

Recall that $\phi \to \psi$ is an abbreviation for $\neg(\phi; \neg\psi)$.

DEFINITION 5.1
$\phi \vdash_\Gamma \psi :\Leftrightarrow$ there are $\phi_1, \dots, \phi_n \in \Gamma$ with $\phi; \neg\neg\phi_1; \cdots; \neg\neg\phi_n \Longrightarrow \psi$. We say that Γ is consistent with ϕ if there is a ψ with $\phi \not\vdash_\Gamma \psi$.

We call Γ negation complete with respect to ϕ if for every ψ either $\phi \vdash_\Gamma \psi$ or $\phi \vdash_\Gamma \neg\psi$.

Call a formula ψ fixed in v if $v \in \mathit{fix}(\psi)$. Γ has witnesses for ϕ if for every formula ψ fixed in v such that $\phi \vdash_\Gamma \exists v; \psi$ there is a c for which $\neg\neg(\exists v; \psi) \rightarrow [c/v]\psi \in \Gamma$.

PROPOSITION 5.2
If $\phi \not\vdash_\Gamma \psi$ then at least one of $\Gamma \cup \{\psi\}$, $\Gamma \cup \{\neg\psi\}$ is consistent with ϕ.

PROOF. Use the Cases Rule. ■

Note that the cases rule uses the test axiom and the following rules of the calculus: left and right $\neg$ rules, test swap rule, right sequencing rule, left and right double negation rules and transitivity rule.

Let $\chi_1, \ldots$ be a list of all formulas of L that are fixed in v. Let $C_0 := c_1^0, \ldots$ be a list of fresh individual constants. Let L_0 be $L(C_0)$ (the result of adding the constants C_0 to L).

$$\Delta_0 := \{\neg\neg(\exists v; \chi_i) \rightarrow [c_i^0/v]\chi_i \mid 1 \leq i\}.$$

Let $\chi_1^m, \ldots$ be a list of all formulas that are fixed in v which occur in L_m. Let $C_{m+1} := c_1^{m+1}, \ldots$ be a list of fresh individual constants. Let $L_{m+1} := L_m(C_{m+1})$.

$$\Delta_{m+1} := \{\neg\neg(\exists v; \chi_i^{m+1}) \rightarrow [c_i^{m+1}/v]\chi_i^{m+1} \mid 1 \leq i\}.$$

Let $C := \bigcup_m C_m$, and let Δ be the set of $L(C)$ formulas given by:

$$\Delta := \bigcup_m \Delta_m.$$

PROPOSITION 5.3
If Γ consists of $L(C)$ formulas, and $\Gamma \supseteq \Delta$, then Γ has witnesses for ϕ.

PROOF. Take some ψ fixed in v with $\phi \vdash_\Gamma \exists v; \psi$. Then $\psi \in L_m$ for some m. So there is some $c \in C$ with $\neg\neg(\exists v; \psi) \rightarrow [c/v]\psi \in \Delta_{m+1}$. So $\neg\neg(\exists v; \psi) \rightarrow [c/v]\psi \in \Delta \subseteq \Gamma$. ■

PROPOSITION 5.4
If Γ is consistent with ϕ then there is a $\Gamma' \supseteq \Gamma$ which is consistent with ϕ, negation complete with respect to ϕ, and has witnesses for ϕ.

PROOF. Assume Γ consistent with ϕ. Let $\chi_1, \ldots, \chi_i, \ldots$ be an enumeration of all formulas of the language $L(C)$. Extend Γ as follows to a Γ' with the required properties.

$$\Gamma_0 := \Gamma \cup \Delta$$

$$\Gamma_{m+1} := \begin{cases} \Gamma_m \cup \{\chi_m\} & \text{if } \Gamma_m \cup \{\chi_m\} \text{ consistent with } \phi, \\ \Gamma_m & \text{otherwise.} \end{cases}$$

$$\Gamma' := \bigcup_m \Gamma_m$$

$\Gamma' \supseteq \Delta$, so by Proposition 5.3 Γ' has witnesses for ϕ.

Assume Γ' is inconsistent with ϕ. Then some Γ_m has to be inconsistent with ϕ and contradiction with Proposition 5.2. So Γ' is consistent with ϕ.

Finally, Γ' is negation complete by construction. ■

Definition 5.5 (Canonical Model)
Let Γ be consistent with ϕ, be negation complete with respect to ϕ, and have witnesses for ϕ. Then $\mathcal{M}_\Gamma = (D, I)$ is defined as follows. $D :=$ the set of variables V together with the set of constants C occurring in $\Gamma \cup \{\phi\}$. For all terms of the language, let $I(t) := t$. Let $I(P) := \{\langle t_1, \dots t_k\rangle \mid \phi \vdash_\Gamma P(t_1, \dots, t_k)\}$.

Lemma 5.6 (Satisfaction Lemma)
Let Γ be consistent with ϕ, be negation complete with respect to ϕ, and have witnesses for ϕ. Then for all ξ: $\phi \vdash_\Gamma \xi$ iff $\exists s$ with $i[\![\xi]\!]^{\mathcal{M}_\Gamma} s$, where i is the identity assignment.

Proof. Induction on the structure of ξ.

$\phi \vdash_\Gamma \top$, for we have:

$$\dfrac{\dfrac{}{\top \Longrightarrow \top}\ \text{test axiom}}{\phi \Longrightarrow \top}\ ;\ \text{left}$$

If $\phi \vdash_\Gamma \exists v; \xi'$ then by modus ponens, using the fact that Γ has witnesses, $\phi \vdash_\Gamma [c/v]\xi'$. By i.h., there is an s with $i[\![[c/v]\xi']\!]^{\mathcal{M}_\Gamma} s$, and therefore $i[\![\exists v; \xi']\!]^{\mathcal{M}_\Gamma} s$. Conversely, if there is an s with $i[\![\exists v; \xi']\!]^{\mathcal{M}_\Gamma} s$ then there is a term t with $i[\![[t/v]\xi']\!]^{\mathcal{M}_\Gamma} s$ (this is because the domain of $\mathcal{M}_\Gamma$ consists of terms). By i.h. from this: $\phi \vdash_\Gamma [t/v]\xi'$. By the quantifier intro rule, $\phi \vdash_\Gamma \exists v; \xi'$.

$\phi \vdash_\Gamma Pt_1 \cdots t_n; \xi'$ iff $\phi \vdash_\Gamma Pt_1 \cdots t_n$ and $\phi \vdash_\Gamma \xi'$ iff $\langle [\![t_1]\!]_i^{\mathcal{M}_\Gamma}, \dots, [\![t_n]\!]_i^{\mathcal{M}_\Gamma}\rangle \in I(P)$ and $\phi \vdash_\Gamma \xi'$ iff $i[\![Pt_1 \cdots t_n]\!]^{\mathcal{M}_\Gamma} i$ and (i.h.) there is an s with $i[\![\xi']\!]^{\mathcal{M}_\Gamma} s$ iff there is an s with $i[\![Pt_1 \cdots t_n; \xi']\!]^{\mathcal{M}_\Gamma} s$.

$\phi \vdash_\Gamma \neg\xi_1; \xi_2$ iff $\phi \vdash_\Gamma \neg\xi_1$ and $\phi \vdash_\Gamma \xi_2$ iff (i.h. twice) $i[\![\neg\xi_1]\!]^{\mathcal{M}_\Gamma} i$ and there is an s with $i[\![\xi_2]\!]^{\mathcal{M}_\Gamma} s$ iff there is an s with $i[\![\neg\xi_1; \xi_2]\!]^{\mathcal{M}_\Gamma} s$. ∎

The case of $\exists v; \xi'$ in the satisfaction lemma uses the quantifier intro rule.

For our next step, we need a small lemma.

Lemma 5.7 (Recycling Lemma)
For all $\mathcal{M}$, all assignments s, r for $\mathcal{M}$, all formulas ϕ, ψ: If $s[\![\exists w; [w/v]\phi; \exists v; \psi]\!]^{\mathcal{M}} r$, then $s[\![\exists v; \phi; \exists v; \psi]\!]^{\mathcal{M}} r$.

Proof. Dropping the $\mathcal{M}$ superscript to avoid clutter, we get from

$$s[\![\exists w; [w/v]\phi; \exists v; \psi]\!] r$$

that there are s', r', r'' with $s \sim_w s'$, $s'[\![[w/v]\phi]\!]r'$, $r' \sim_v r''$ and $r''[\![\psi]\!]r$. Now there are two cases.

In case $v \in \mathit{intro}(\phi)$, we get by Lemma 2.6 from $s'[\![[w/v]\phi]\!]r'$ that

$$s'[v := [\![w]\!]_{s'}][\![\phi]\!]r',$$

and therefore

$$s[\![\exists v; \phi]\!]r' \text{ and } r'[\![\exists v; \psi]\!]r,$$

and done.

In case $v \notin \mathit{intro}(\phi)$, we get by Lemma 2.6 from $s'[\![[w/v]\phi]\!]r'$ that

$$s'[v := [\![w]\!]_{s'}][\![\phi]\!]r'[v := [\![w]\!]_{s'}],$$

and therefore

$$s[\![\exists v;\phi]\!]r'[v := [\![w]\!]_{s'}] \text{ and } r'[v := [\![w]\!]_{s'}][\![\exists v;\psi]\!]r,$$

and done. ∎

PROPOSITION 5.8
Let Γ be consistent with ϕ, be negation complete with respect to ϕ, and have witnesses for ϕ. Then there is an s with $s[\![\phi]\!]^{\mathcal{M}_\Gamma} i$, where i is the identity assignment.

PROOF. Let ϕ' be the result of replacing all patterns of the form $\exists v; \psi; \exists v$ in ϕ by patterns $\exists w; [w/v]\psi; \exists v$, with w fresh.

Next, let $\exists v_1; \cdots; \exists v_k; \phi''$ be the result of moving all active (top level) quantifiers of ϕ' to the front.

Now ϕ'' is a test, and we have:

$$\cfrac{\cfrac{\overline{\phi'' \Longrightarrow \phi''}\ \text{test axiom}}{\exists v_1; \cdots; \exists v_k; \phi'' \Longrightarrow \phi''}\ ;\ \text{left}}{\phi \Longrightarrow \phi''}\ \text{quantifier swaps} + \text{variable refreshments}$$

Therefore, $\phi \vdash_\Gamma \phi''$, and we get from the satisfaction lemma, plus the fact that ϕ'' is a test, that $i[\![\phi'']\!]^{\mathcal{M}_\Gamma} i$. Also, by definition of the semantics for $\exists v$, we have that there is an s with $s[\![\exists v_1; \cdots; v_k; \phi'']\!]^{\mathcal{M}_\Gamma} i$. Since quantifier swaps do not affect the relational semantics, $s[\![\phi']\!]^{\mathcal{M}_\Gamma} i$. Since the quantifier refreshments needed for the transformation from ϕ' to ϕ are of the format described in the Recycling Lemma, we get from this lemma that $s[\![\phi]\!]^{\mathcal{M}_\Gamma} i$. ∎

This uses the test axiom, the variable refreshment rule and the test-quantifier swap rules.

THEOREM 5.9 (Completeness)
If $\phi \models \psi$ then $\phi \Longrightarrow \psi$.

PROOF. Assume $\phi \not\Longrightarrow \psi$. Then $\neg\psi$ is consistent with ϕ. By proposition 5.4, there is a $\Gamma \supseteq \{\neg\psi\}$ which is consistent with ϕ, is negation complete with respect to ϕ, and has witnesses for ϕ. Construct the canonical model and apply the satisfaction lemma to get:

$$i[\![\neg\psi]\!]^{\mathcal{M}_\Gamma} i.$$

By the semantic clause for negation we have that for all r:

$$\text{not } i[\![\psi]\!]^{\mathcal{M}_\Gamma} r.$$

By proposition 5.8 there is an s with

$$s[\![\phi]\!]^{\mathcal{M}_\Gamma} i.$$

This proves $\phi \not\models \psi$. ∎

A simple inspection confirms that all the ingredients of the calculus were used in the completeness proof.

6 Anaphoric Reasoning with Equality

Anaphoric linking makes extensive use of equality. See Van Eijck [2] for an in-depth analysis of the use of equality in anaphoric descriptions. An anaphoric definite description like *the garden* can be treated as a definiteness quantifier followed by a link to a contextually available referent. The translation of *He sprinkles the garden* would then be something like $\iota x : (x \doteq y; Gx); Szx$. Also, the determiner *another* often has an implicit anaphoric element. In such cases, the treatment involves non-identity links to contextually available referents. *He met another woman* gets a translation like $\exists x; x \neq y; Wx; Mzx)$. Below we indicate briefly how to handle equality, while leaving the axiomatisation of definiteness in the present framework for another occasion.

For the treatment of equality, add expressions $t_1 \doteq t_2$ to the language (we assume that $t_1 \neq t_2$ is an abbreviation of $\neg(t_1 \doteq t_2)$). Equalities are tests, or, in other words, $\mathit{intro}(t_1 \doteq t_2) = \emptyset$. The semantics of equality is as you would expect:

$$s[\![t_1 \doteq t_2]\!]^{\mathcal{M}} r \quad \text{iff} \quad s = r \text{ and } [\![t_1]\!]^{\mathcal{M}}_s \text{ equals } [\![t_2]\!]^{\mathcal{M}}_s.$$

To deal with equality statements, a reflexivity axiom and a substitution rule must be added to the calculus.

Reflexivity Axiom

$$\frac{}{\phi \Longrightarrow t \doteq t}$$

Soundness of Reflexivity Axiom

The axiom expresses that equality is reflexive.

Substitution Rule

For this we need the notion $[t_1/t_2]$ of substitution of terms for arbitrary terms. This is defined in the obvious way.

$$\frac{\phi \Longrightarrow [t_1/t_2]\psi}{\phi; t_1 \doteq t_2 \Longrightarrow \psi}\ t_1, t_2 \in C \cup \mathit{intro}(\phi)$$

Soundness of the Substitution Rule

Assume $s[\![\phi; t_1 \doteq t_2]\!]^{\mathcal{M}} r$. Then $s[\![\phi]\!]^{\mathcal{M}} r$, and $[\![t_1]\!]^{\mathcal{M}}_r = [\![t_2]\!]^{\mathcal{M}}_r$. By the soundness of the premiss, there is an r' with $r[\![[t_1/t_2]\psi]\!]^{\mathcal{M}} r'$. Therefore, $r[\![\psi]\!]^{\mathcal{M}} r'$. This shows $\phi; t_1 \doteq t_2 \models \psi$.

This is enough to derive the symmetry and transitivity of equality. To see this, here are the relevant derivations:

$$\cfrac{\cfrac{}{\top \Longrightarrow t_1 \doteq t_1}\ \text{refl}}{t_1 \doteq t_2 \Longrightarrow t_2 \doteq t_1}\ \text{subst}$$

The application of the substitution rule is correct, for $t_1 \doteq t_1$ is of the form $[t_1/t_2]t_2 \doteq t_1$.

Here is the derivation of the transitivity rule:

$$\dfrac{\overline{t_1 \doteq t_2 \Longrightarrow t_1 \doteq t_2}\ \text{test}}{t_1 \doteq t_2; t_2 \doteq t_3 \Longrightarrow t_1 \doteq t_3}\ \text{subst}$$

To see that the application of the substitution rule is correct, notice that $t_1 \doteq t_2$ is of the form $[t_2/t_3]t_1 \doteq t_3$.

The completeness of the anaphoric DPL calculus with equality is proved by modifying the Henkin construction in the usual way, taking equivalence classes of terms under provable equality as elements of the canonical model. The canonical valuation i is now given by $i(v) := [v]$, where $[v]$ is the equivalence class containing v.

7 A Variation on the Theme

To justify the plural in the title of this paper, first note that the approach to axiomatising dynamic predicate logic carries over without a hitch to the analysis of Discourse Representation Theory [8, 5]. The version for DRT looks even more elegant because the shift rules are absorbed by the set theoretic notation for DRSs. This reflects the fact that the syntax of DRT is more abstract than that of DPL.

To keep as close as possible to the previous calculus, we consider a version of DRT where DRS negation is primitive and $D_1 \Rightarrow D_2$ is defined in terms of negation. We can view a Discourse Representation Structure or DRS as a triple consisting of a set of fixed referents F, a set of introduced referents I, and a set of conditions $C_1 \cdots C_n$ constrained by the requirement that the free variables in the C_i must be among $F \cup I$.

Concretely, the syntax of DRT looks like this:

Definition 7.1 (DRT)

$$\begin{array}{rcl} t & ::= & c \mid v \\ C & ::= & \top \mid Pt_1 \cdots t_n \mid \neg D \\ D & ::= & \begin{array}{|c|c|} \hline v_1 \cdots v_n & v_{n+1} \cdots v_m \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_k} \\ \hline \end{array} \end{array}$$

The (active) markers of a term t or condition C or DRS D are given by:

$$M(v) := \{v\}, M(c) := \emptyset, M(\top) := \emptyset, M(Pt_1 \cdots t_n) := \cup_i M(t_i),$$

$$M(\begin{array}{|c|c|} \hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}) := F \cup I, M(\neg \begin{array}{|c|c|} \hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}) := F.$$

Conditions on the formation rule for a DRS $\begin{array}{|c|c|} \hline v_1 \cdots v_n & v_{n+1} \cdots v_m \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_k} \\ \hline \end{array}$:

1. $\{v_1 \cdots v_n\} \cap \{v_{n+1} \cdots v_m\} = \emptyset$,
2. $\cup_i M(C_i) \subseteq \{v_1 \cdots v_m\}$.

We can now define the condition

$$\begin{array}{|c|c|} \hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline \end{array} \Rightarrow D$$

FIG. 2. The Calculus for DRT

test axiom

$$\frac{}{\begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \; M(C) \subseteq F$$

transitivity

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}$$

marker intro

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{[t/v]\ C} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & \{v\} \cup I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \; v \notin F$$

marker shift

$$\frac{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}{\begin{array}{|c|c|}\hline F - I_0 & I_0 \cup I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}$$

sequencing-l

$$\frac{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_0 \cup C} \\ \hline \end{array} \Longrightarrow D} \; M(C_0) \subseteq F \cup I$$

sequencing-r

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \qquad D \Longrightarrow \begin{array}{|c|c|}\hline F & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & I_1 \cup I_2 \\ \hline \multicolumn{2}{|c|}{C_1 \cup C_2} \\ \hline \end{array}} \; I_1 \cap I_2 = \emptyset$$

negation

$$\frac{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array} \Longrightarrow D}{\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C \cup \{\neg D\}} \\ \hline \end{array} \Longrightarrow \bot} \qquad \frac{\begin{array}{|c|c|}\hline F & I_1 \cup I_2 \\ \hline \multicolumn{2}{|c|}{C_1 \cup C_2} \\ \hline \end{array} \Longrightarrow \bot}{\begin{array}{|c|c|}\hline F & I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F \cup I_1 & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F \cup I_1 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}} \\ \hline \end{array}} \; M(C_1) \cap I_2 = \emptyset$$

d-negation

$$\frac{D \Longrightarrow \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \\ \hline \end{array}} \\ \hline \end{array}}{D \Longrightarrow \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}} \qquad \frac{\begin{array}{|c|c|}\hline F & I_1 \\ \hline \multicolumn{2}{|c|}{C_1 \cup \{\neg \begin{array}{|c|c|}\hline F \cup I_1 & \emptyset \\ \hline \multicolumn{2}{|c|}{\neg \begin{array}{|c|c|}\hline F \cup I_1 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}} \\ \hline \end{array}\}} \\ \hline \end{array} \Longrightarrow \bot}{\begin{array}{|c|c|}\hline F & I_1 \cup I_2 \\ \hline \multicolumn{2}{|c|}{C_1 \cup C_2} \\ \hline \end{array} \Longrightarrow \bot}$$

as

$$\neg \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n, \neg D} \\ \hline \end{array}.$$

Here is a semantics in terms of partial assignments, following the original set-up in [8].

DEFINITION 7.2 (Semantics of DRT)

$$\begin{array}{lll}
\mathcal{M}, f \models \top & & \text{always} \\
\mathcal{M}, f \models Pt_1 \cdots t_n & \text{iff} & \langle [\![t_1]\!]_f^{\mathcal{M}}, \dots, [\![t_n]\!]_f^{\mathcal{M}} \rangle \in P^{\mathcal{M}} \\
\mathcal{M}, f \models \neg D & \text{iff} & \text{there is no } g \text{ with } \mathcal{M}, f, g \models D \\
 & & \\
\mathcal{M}, f, g \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline \end{array} & \text{iff} & \begin{array}{l} f : F \to dom(\mathcal{M}), \\ g : F \cup I \to dom(\mathcal{M}), f = g \upharpoonright F, \\ M, g \models C_1, \ \dots \ , M, g \models C_n. \end{array}
\end{array}$$

DEFINITION 7.3 (DRT Consequence)

$$\begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline \end{array} \models \begin{array}{|c|c|}\hline F \cup I & I' \\ \hline \multicolumn{2}{|c|}{C_{n+1} \cdots C_m} \\ \hline \end{array}$$

iff for all $\mathcal{M}, f, g$ with $\mathcal{M}, f, g \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline \end{array}$
there is an h with $\mathcal{M}, g, h \models \begin{array}{|c|c|}\hline F \cup I & I' \\ \hline \multicolumn{2}{|c|}{C_{n+1} \cdots C_m} \\ \hline \end{array}$.

A DRT calculus along similar lines as the calculus for DPL above is given in Figure 2 (lists $C_1 \cdots C_k$ abbreviated as C). The calculus uses substitution in constraints and DRSs. This notion is defined by:

$$\begin{array}{rcl}
[t/v] \ Pt_1 \cdots t_n & := & P[t/v]t_1 \cdots [t/v]t_n, \\
[t/v] \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_n} \\ \hline \end{array} & := & \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{[t/v] \ C_1 \cdots [t/v] \ C_n} \\ \hline \end{array}
\end{array}$$

Of course, when a substitution $[t/v]$ is mentioned in a rule, it is assumed that t is free for v in D. It is also assumed that all DRSs mentioned in the rules satisfy the syntactic wellformedness conditions for DRSs.

Note that the counterpart to variable refreshment, marker refreshment, is not among the rules of the DRT calculus. Marker refreshment is the following rule:

$$\frac{\begin{array}{|c|c|}\hline F_1 & \{v\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{\begin{array}{|c|c|}\hline F_1 & \{w\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{[w/v]C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline \{w\} \cup F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{[w/v] \ C_2} \\ \hline \end{array}}$$

In fact, this is an admissible rule of the calculus, in the sense that adding the rule would not change the set of valid DRT sequences. To see why this is so, note that a proof for

$$\begin{array}{|c|c|}\hline F_1 & \{v\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}$$

can be transformed into a proof for

$$\begin{array}{|c|c|}\hline F_1 & \{w\} \cup I_1 \\ \hline \multicolumn{2}{|c|}{[w/v]\ C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline \{w\} \cup F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{[w/v]\ C_2} \\ \hline \end{array}$$

by simply changing appropriate occurrences of v to w. Note that the steps of this proof transformation are not rules of the calculus. Rather, the procedure of proof transformation is an argument to establish that for any proof using applications of marker refreshment an alternative proof can be found that does not use marker refreshment.

Note that marker shift and sequencing left together give the following derived rule:

$$\frac{\begin{array}{|c|c|}\hline F_1 & I_1 \\ \hline \multicolumn{2}{|c|}{C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}{\begin{array}{|c|c|}\hline F_1 - I_0 & I_0 \cup I_1 \\ \hline \multicolumn{2}{|c|}{C_0 \cup C_1} \\ \hline \end{array} \Longrightarrow \begin{array}{|c|c|}\hline F_2 & I_2 \\ \hline \multicolumn{2}{|c|}{C_2} \\ \hline \end{array}}$$

THEOREM 7.4
The calculus for DRT is sound.

PROOF. Induction on the basis that the test axiom is sound and that the rules preserve soundness. Here is one example soundness check, for the rule of marker introduction. Assume $\mathcal{M}, f, g \models D$. Then by the soundness of the premiss, there is an h with $\mathcal{M}, g, h \models \begin{array}{|c|c|}\hline F & I \\ \hline \multicolumn{2}{|c|}{[t/v]\ C} \\ \hline \end{array}$. Thus, $\mathcal{M}, h \models [t/v]\ C$. Since $v \notin F$, h' given by $h'(v) = [\![t]\!]_h^{\mathcal{M}}$, and $h'(w) = h(w)$ for all $w \neq v$ for which h is defined extends g. By (an appropriate DRT version of) the substitution lemma, $\mathcal{M}, g, h' \models \begin{array}{|c|c|}\hline F & \{v\} \cup I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}$. This proves $D \models \begin{array}{|c|c|}\hline F & \{v\} \cup I \\ \hline \multicolumn{2}{|c|}{C} \\ \hline \end{array}$. ∎

THEOREM 7.5
The calculus for DRT is complete.

PROOF. Consider the following translation procedure from DRT to DPL:

$$\begin{array}{rcl} \top^\circ & := & \top \\ (Pt_1 \cdots t_n)^\circ & := & Pt_1 \cdots t_n \\ (\neg D)^\circ & := & \neg D^\bullet \end{array}$$

$$\begin{array}{|c|c|}\hline v_1 \cdots v_n & v_{n+1} \cdots v_m \\ \hline \multicolumn{2}{|c|}{C_1 \cdots C_k} \\ \hline \end{array}^{\bullet} \quad := \quad \exists v_{n+1}; \cdots ; \exists v_m; C_1^\circ; \cdots ; C_k^\circ$$

Use this procedure to translate a DRT sequent $D_1 \Longrightarrow D_2$ to a DPL sequent $D_1^\bullet \Longrightarrow D_2^\bullet$. Note that the translation maps valid DRT consequences to valid DPL consequences. Assume $D_1 \models D_2$. Then $D_1^\bullet \models D_2^\bullet$. By Theorem 5.9, $D_1^\bullet \Longrightarrow D_2^\bullet$. Next, transform the proof of $D_1^\bullet \Longrightarrow D_2^\bullet$ into one of $D_1 \Longrightarrow D_2$ making use of the correspondences between the axioms and rules of the two calculi. (Note that marker shift corresponds to sequencing left with a list of quantifiers.) ∎

8 A Further Variation

Here is a variation on dynamic predicate logic suggested by Albert Visser in [13]. Visser proposes the following 'prudent' version of DPL. States are pairs (X, s) of a variable set X and an assignment s. The semantics of prudent DPL is given by:

Definition 8.1 (Semantics of Prudent DPL)

$(X,s)[\![\top]\!]^{\mathcal{M}}(Y,r)$	iff	$(X,s) = (Y,r)$
$(X,s)[\![\exists v;\psi]\!]^{\mathcal{M}}(Y,r)$	iff	(i) $v \in X$ and $(X,s)[\![\psi]\!]^{\mathcal{M}}(Y,r)$, or (ii) $v \notin X$ and $(X \cup \{v\}, s')[\![\psi]\!]^{\mathcal{M}}(Y,r)$, where $s' \sim_v s$
$(X,s)[\![Pt_1 \cdots t_n;\psi]\!]^{\mathcal{M}}(Y,r)$	iff	$\langle [\![t_1]\!]^{\mathcal{M}}_s, \ldots, [\![t_n]\!]^{\mathcal{M}}_s \rangle \in P^{\mathcal{M}}$ and $(X,s)[\![\psi]\!]^{\mathcal{M}}(Y,r)$
$(X,s)[\![\neg(\phi);\psi]\!]^{\mathcal{M}}(Y,r)$	iff	there is no (Y',r') with $(X,s)[\![\phi]\!]^{\mathcal{M}}(Y',r')$, and $(X,s)[\![\psi]\!]^{\mathcal{M}}(Y,r)$.

Here the set of 'activated' variables is incorporated in the relational semantics. This set is used in the semantic clause for the existential quantifier. In case variable v has already been activated before, $\exists v$ does nothing.

Semantic consequence for prudent DPL gets defined as $\phi \models \psi$ iff for all models $\mathcal{M}$, all assignments s, r and the variable set X with $(\emptyset, s)[\![\phi]\!]^{\mathcal{M}}(X,r)$ there are (X', r') with $(X,r)[\![\psi]\!]^{\mathcal{M}}(X',r')$.

The semantic change gets reflected in the calculus as follows:

- The restriction $v \notin \mathit{fix}(T)$ on test-quantifier swap (from $T;\exists v$ to $\exists v;T$) and quantifier-test swap (from $\exists v;T$ to $T;\exists v$) are relaxed to: either $v \in \mathit{intro}(C_1)$ or $v \notin \mathit{fix}(T)$, where C_1 is the left context.
- The rule of variable refreshment is removed from the calculus.

It is easily seen that the rule of variable refreshment is not sound for the prudent semantics. It can be proved that this modified calculus is sound and complete for prudent DPL. The fact that the modified calculus is a (slight) simplification of the original can be taken as a (slight) point in favour of prudent DPL.

9 Related Work

The present approach to axiomatising dynamic predicate logics derives from a calculus for dynamic reasoning without variables [6] where variables have been replaced by De Bruyn indices (familiar from implementations of lambda calculus), and destructive assignment is replaced by a stack push operation. Such a variable free approach to dynamics has clear advantages over the destructive dynamics of variable assignment if one is interested in incrementality of updates.

An alternative to the present calculus is given in unpublished work by Frank Veltman [12]. Veltman analyses a more baroque version of DPL, where a distinction is made between the 'text connective' ; and the 'sentential connective' $\wedge$. Their semantic interpretations are the same; the difference between them resides in a syntactic restriction on the text connective to the effect that it cannot be outscoped by negation. Also, in Veltman's system free text variables are forbidden on the grounds that these are hard to analyse in terms of partial variable assignments.

Veltman's central notion of substitution is defined for variables that are 'free in ϕ after ψ'. In our terminology, an occurrence of v in ϕ is free in ϕ after ψ if $v \in \mathit{fix}(\phi)$ and $v \notin \mathit{intro}(\psi)$ both hold. Because Veltman's calculus does not have our division of labour between introduction and elimination rules on one hand and shift rules on the other, the introduction and elimination rules have rather involved side conditions. Veltman suggests that completeness of these rules can be proved by means of a syntactic tour through a variety of equivalent tableaux calculi, but [12] contains only a very sketchy roadmap for carrying this out.

The calculus for DRT given in Section 7 demonstrates the merits of keeping track both of the set of fixed discourse referents and of the set of introduced discourse referents of a DRS. This distinction allows for a particulary simple formulation of the transitivity rule, while it also considerably simplifies the formulation of the side conditions on some of the other rules. How the present calculus compares to previous calculi for DRT as regards simplicity is a matter of taste that we leave to the reader (compare with e.g. [9, 10]). The present approach is very much in the spirit of Visser [13]. At least one merit can be claimed for the present approach without getting entangled in a dispute about taste: the present approach makes the relationship between reasoning in DRT and reasoning in related formalisms (such as DPL and prudent DPL) much clearer than previous DRT calculi ever did.

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