Update, Probability, Knowledge and Belief

Jan van Eijck[1,2] and Bryan Renne[3]

[1]Centrum Wiskunde & Informatica (CWI)
Amsterdam

[2]Institute for Logic, Language and Computation (ILLC)
Amsterdam

[3]University of British Columbia, Faculty of Medicine
Vancouver, Canada

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Abstract

The paper compares two kinds of models for logics of knowledge and belief, neighbourhood models and epistemic weight models. We give sound and complete calculi for both, and we show that our calculus for neighbourhood models is sound but not complete for epistemic weight models. Epistemic weight models combine knowledge and probability by using epistemic accessibility relations and weights to define subjective probabilities. Our Probability Comparison Calculus for this class of models is a further simplification of the calculus that was presented in AIML 2014.

Outline

Probability and Information

Outline

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Epistemic Neighbourhood Models

Outline

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Laplace on Causes of Disagreement Between People

When concerned with things that are only likely true, the difference in how informed every man is about them is one of the principal causes of the diversity of opinions about the same objects.

Combining DEL and Probability

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- Prehistory of this: De Finetti [Fin37, Fin51].

De Finetti's Requirements for Qualitative Probability

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nonnegativity	$A \succeq \emptyset$
nontriviality	$\emptyset \not\succeq W$
totality	$A \succeq B$ or $B \succeq A$
transitivity	if $A \succeq B$ and $B \succeq C$ then $A \succeq C$
quasi-additivity	if $(A \cup B) \cap C = \emptyset$ then $A \succeq B$ iff $A \cup C \succeq B \cup C$

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- A probability measure on W is a function $\mu : \mathcal{P}(W) \to \mathbb{R}$ satisfying $\mu(\emptyset) = 0$, $\mu(W) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A, B \subseteq W$ with $A \cap B = \emptyset$ (additivity).

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- De Finetti's conjecture: the five requirements completely determine a probability measure on W.

De Finetti's Conjecture Refuted

Theorem

There is a relation satisfying De Finetti's axioms that does not agree with any probability measure [KPS59].

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- Consider $W = \{p, q, r, s, t\}$ with a weight map $\nu : W \to \mathbb{N}$ given by $\nu(p) = 4, \nu(q) = 1, \nu(r) = 3, \nu(s) = 2, \nu(t) = 6$.

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- Define $\succeq$ as

$$\succeq := \succeq_\nu - \{(st, pqr)\}.$$

  This yields: $p \approx qr, rs \approx pq, qt \approx pr, pqr \succ st$, and $\succeq$ satisfies the De Finetti axioms.

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This yields: $p \approx qr, rs \approx pq, qt \approx pr, pqr \succ st$, and $\succeq$ satisfies the De Finetti axioms.
- $\succeq$ does not agree with any probability measure μ:
- It follows from $\mu(p) = \mu(qr)$, $\mu(rs) = \mu(pq)$, $\mu(qt) = \mu(pr)$ that $\mu(st) = \mu(pqr)$. Thus, μ cannot agree with $pqr \succ st$.

Scott Axioms for $\succeq$

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- A pair of k-length sequences of sets $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ is *balanced* if for each $w \in W$ it holds that $|\{i \mid w \in A_i\}| = |\{i \mid w \in B_i\}|$.

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- The Scott axiom for $\succeq$ for length k (k-cancellation):

  if $(A_1, \ldots, A_k, X)$ and $(B_1, \ldots, B_k, Y)$ are balanced, and $A_i \succeq B_i$ for each i with $1 \leq i \leq k$, then $Y \succeq X$.

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- If a relation $\succeq$ is representable by a probability measure, then $\succeq$ must satisfy cancellation for any k.
- Scott [Sco64]: any $\succeq$ relation satisfying nonnegativity, nontriviality, totality and cancellation for any $k \in \mathbb{N}$ determines a probability measure.

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Representing Belief as Truth in a Neighbourhood

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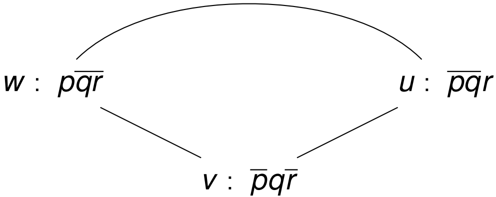
Neighbourhood Belief Not Closed Under Conjunction



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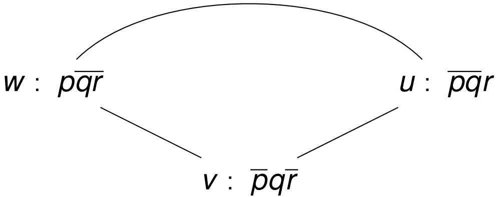

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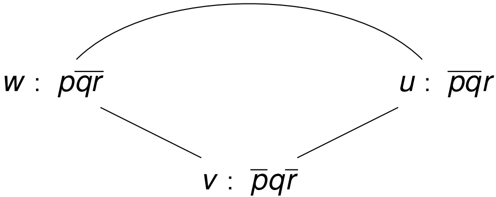

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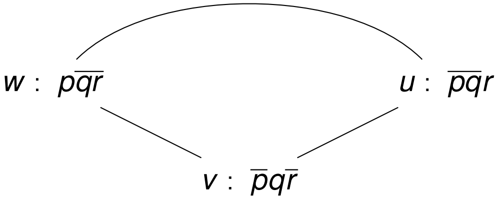

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- In all worlds $B(\neg p \wedge \neg q)$, $B(\neg p \wedge \neg r)$, $B(\neg q \wedge \neg r)$ are false.

ED Calculus for Epistemic Neighbourhood Logic

(Taut)	All instances of propositional tautologies
(Dist-K)	$K_i(\phi \to \psi) \to K_i\phi \to K_i\psi$
(T)	$K_i\phi \to \phi$
(PI-K)	$K_i\phi \to K_iK_i\phi$
(NI-K)	$\neg K_i\phi \to K_i\neg K_i\phi$
(N)	$B_i\top$.
(PI-KB)	$B_i\phi \to K_iB_i\phi$
(NI-KB)	$\neg B_i\phi \to K_i\neg B_i\phi$
(M)	$K_i(\phi \to \psi) \to B_i\phi \to B_i\psi$
(D)	$B_i\phi \to \check{B}_i\phi$.
(SC)	$\check{B}_i\phi \wedge \check{K}_i(\neg\phi \wedge \psi) \to B_i(\phi \vee \psi)$

$$\frac{\phi \to \psi \quad \phi}{\psi}\ \text{(MP)} \qquad \frac{\phi}{K_i\phi}\ \text{(Nec-K)}$$

Soundness and Completeness

Theorem

ED calculus is sound and complete for Epistemic Neighbourhood Models.

Epistemic Weight Models

An **epistemic weight model** for agents I and basic propositions P is a tuple $\mathcal{M} = (W, R, L, V)$ where

- W is a non-empty countable set of worlds,
- R assigns to every agent $i \in I$ an equivalence relation $\sim_i$ on W,
- L assigns to every $i \in I$ a function $\mathbb{L}_i$ from W to $\mathbb{Q}^+$ (the positive rationals), subject to the following boundedness condition (*).

$$\forall i \in I \forall w \in W \sum_{u \in [w]_i} \mathbb{L}_i(u) < \infty. \qquad (*)$$

where $[w]_i$ is the cell of w in the partition induced by $\sim_i$.
- V assigns to every $w \in W$ a subset of P,

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Truth in Weight Models

- Use $\mathbb{L}_i(X)$ for $\sum_{x \in X} \mathbb{L}_i(x)$.
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- **Theorem**
  ED calculus is sound for epistemic weight models.

CWI

Agreement, Incompleteness

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- Definition (Agreement)

  Let $\mathcal{M} = (W, R, N, V)$ be a neighbourhood model and let L be a weight function for $\mathcal{M}$. Then L *agrees with* $\mathcal{M}$ if it holds for all agents i and all $w \in W$ that

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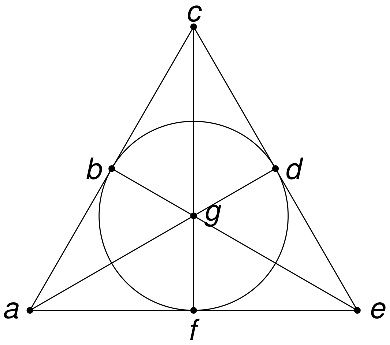
- Theorem

  There exists an epistemic neighbourhood model $\mathcal{M}$ that has no agreeing weight function.

Incompleteness: Example from the Fano plane



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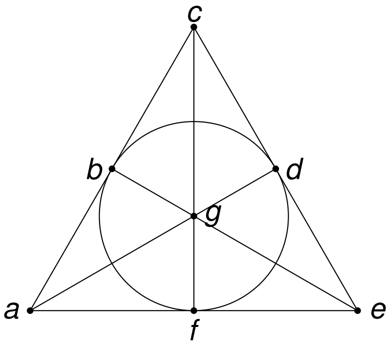

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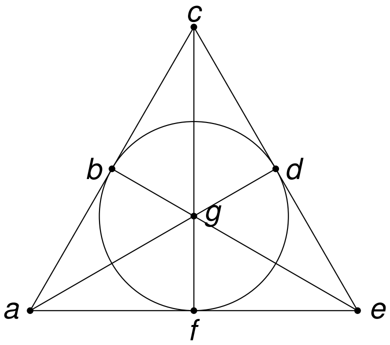

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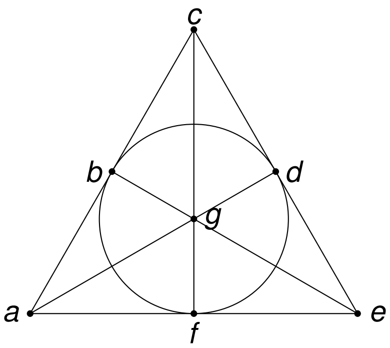

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- Contradiction. So no such $\mathbb{L}$ exists.

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Definition (EC Language)

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Truth for EC Logic

Let $\mathcal{M} = (W, R, L, V)$ be an epistemic weight model, let $w \in W$.

$$
\begin{array}{lcl}
[\![\phi]\!]_{\mathcal{M}} & := & \{w \in W \mid \mathcal{M}, w \models \phi\} \\
[\![\phi]\!]^{w,i}_{\mathcal{M}} & := & [\![\phi]\!]_{\mathcal{M}} \cap [w]_i \\
\mathbb{L}_{w,i}\phi & := & \mathbb{L}_i([\![\phi]\!]^{w,i}_{\mathcal{M}}) \\
 & & \\
\mathcal{M}, w \models \top & & \text{always} \\
\mathcal{M}, w \models \neg\phi & \text{iff} & \text{not } \mathcal{M}, w \models \phi \\
\mathcal{M}, w \models \phi_1 \wedge \phi_2 & \text{iff} & \mathcal{M}, w \models \phi_1 \text{ and } \mathcal{M}, w \models \phi_2 \\
\mathcal{M}, w \models \Phi \leq_i \Psi & \text{iff} & \sum_{\phi \in \Phi} \mathbb{L}_{w,i}\phi \leq \sum_{\psi \in \Psi} \mathbb{L}_{w,i}\psi
\end{array}
$$

$\sum_{\phi \in \Phi}$ sums over *occurrences* of ϕ in the list Φ.
Weight function and epistemic accessibility relation together determine probability:

$$
P^{\mathcal{M}}_{w,i}\phi := \frac{\mathbb{L}_{w,i}\phi}{\mathbb{L}_{w,i}\top} \left(= \frac{\mathbb{L}_i([\![\phi]\!]_{\mathcal{M}} \cap [w]_i)}{\mathbb{L}_i([w]_i)} \right)
$$

EC Calculus

Taut	instances of propositional tautologies
ProbT	$(\top \leq_i \phi) \to \phi$
ProbImpl	$\top \leq_i (\phi \to \psi) \to (\phi \leq_i \psi)$
PropPos	$(\Phi \leq_i \Psi) \to \top \leq_i (\Phi \leq_i \Psi)$
PropNeg	$(\Phi >_i \Psi) \to \top \leq_i (\Phi >_i \Psi)$
PropAdd	$(\phi \wedge \psi) \oplus (\phi \wedge \neg\psi) =_i \phi$
Tran	$(\Phi \leq_i \Psi) \wedge (\Psi \leq_i \Xi) \to (\Phi \leq_i \Xi)$
Tot	$(\Phi \leq_i \Psi) \vee (\Psi \leq_i \Phi)$
ComL	$(\Phi_1 \oplus \Phi_2 \leq_i \Psi) \leftrightarrow (\Phi_2 \oplus \Phi_1 \leq_i \Psi)$
ComR	$(\Phi \leq_i \Psi_1 \oplus \Psi_2) \leftrightarrow (\Phi \leq_i \Psi_2 \oplus \Psi_1)$
Add	$(\Phi_1 \leq_i \Psi_1) \wedge (\Phi_2 \leq_i \Psi_2) \to (\Phi_1 \oplus \Phi_2 \leq_i \Psi_1 \oplus \Psi_2)$
Succ	$(\Phi \oplus \top \leq_i \Psi \oplus \top) \to (\Phi \leq_i \Psi)$
MP	From $\vdash \phi$ and $\vdash \phi \to \psi$ derive $\vdash \psi$
NEC	From $\vdash \phi$ derive $\vdash \top \leq_i \phi$

Completeness of EC Calculus

Theorem (Completeness of EC Logic)

The EC calculus is complete for epistemic weight models.

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- From Epistemic Probability Models to Epistemic Neighbourhood Models:
  If $\mathcal{M} = (W, R, L, V)$ is an epistemic weight model, then $\mathcal{M}^\bullet$ is the tuple (W, R, N, V) given by replacing the weight function by a function N, where

$$N_i(w) = \{X \subseteq [w]_i \mid \mathbb{L}_i(X) > \mathbb{L}_i([w]_i - X)\}.$$

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Theorem

For all ED formulas ϕ, for all epistemic probability models $\mathcal{M}$, for all worlds w of $\mathcal{M}$: $\mathcal{M}^\bullet, w \models \phi$ iff $\mathcal{M}, w \models \phi$.

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- **Should you believe you have disease D?**

Weight Model for the Disease Problem

$$\begin{tikzcd} dt\ 0.9 \arrow[r, dash] \arrow[d, dash] & d\bar{t}\ 0.1 \arrow[d, dash] \\ \bar{d}t\ 0.2 * 99 \arrow[r, dash] & \overline{dt}\ 0.8 * 99 \end{tikzcd}$$

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$$P(d) = \frac{0.9}{0.9+0.2*99} = \frac{9}{207} = \frac{1}{23}$$

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  - $\sim_i^{\pm\phi} = \{(w, v) \in W^2 \mid w \sim_i v \text{ and } \mathcal{M}, w \models \phi \text{ iff } \mathcal{M}, v \models \phi\}$.
  - $L^{\pm\phi} = L$,
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- The operation $[\pm\phi]$ cuts the *i*-accessibility links between ϕ and $\neg\phi$ worlds, for all agents *i*.

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Compare with Applying Bayes' Rule

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\neg D)P(\neg D)}$$

$P(T|D) = 0.9, P(D) = 0.01, P(\neg D) = 0.99, P(T|\neg D) = 0.2$
$P(D|T) = \frac{1}{23}$.

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- Extend the logic to capture the distinction between **risk** and **uncertainty** [Kni21].

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