# Common Knowledge in Update Logics

Johan van Benthem*, Jan van Eijck† and Barteld Kooi‡

**Abstract**

Current dynamic epistemic logics often become cumbersome and opaque when common knowledge is added for groups of agents. Still, postconditions regarding common knowledge express the essence of what communication achieves. We present some methods that yield so-called reduction axioms for common knowledge. We investigate the expressive power of public announcement logic with relativized common knowledge, and present reduction axioms that give a detailed account of the dynamics of common knowledge in some major communication types.

## 1 Introduction

Epistemic logic typically deals with what agents consider possible given their current information. This includes knowledge about facts, but also *higher-order information* about information that other agents have. A prime example is *common knowledge*. A formula $\varphi$ is common knowledge if everybody knows $\varphi$, everybody knows that everybody knows that $\varphi$, and so on. How higher order information develops in multi-agent systems due to communication is the focus of many investigations (see for example [1, 5, 10, 13]). Common knowledge is particularly interesting in these contexts, because it is often exactly what communication tries to achieve. Update logics also aim to analyze changes in higher-order information.

In update logics reduction axioms play an important role. They provide easy completeness proofs, they provide expressivity results, and they provide a precise account of what updates achieve by relating what is the case *after* an update to what is the case *before* an update. For instance, the logic of public announcements without common knowledge has an easy completeness proof due to axioms such as $[\varphi]\Box_a\psi \leftrightarrow (\varphi \rightarrow \Box_a[\varphi]\psi)$. This reduction axiom says that knowledge of $\psi$ for agent $a$ is achieved by the announcement that $\varphi$ iff $\varphi$ implies that agent $a$ knows that after the announcement that $\varphi$ it is the case that $\psi$. The completeness proof works by a translation that follows the reduction axioms. Formulas with announcements are translated to provably equivalent ones without announcements. Then completeness follows from completeness of the base logic. This approach is also taken in [6] and [2] for more general epistemic updates.

Reduction axioms are not readily available in update logics with common knowledge, as the logic with updates is more expressive than the logic without them [2]. In [8] relativized common knowledge was introduced to overcome this problem for public announcement logic with common knowledge, and it was also shown that the problem can be overcome with an epistemic Automata

*ILLC, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands & Philosophy Department, Stanford University, `johan@science.uva.nl`

†CWI, P.O. Box 94079 1090 GB Amsterdam, `jve@cwi.nl`

‡Department of Philosophy, University of Groningen, Oude Boteringstraat 52, 9712 GL Groningen, The Netherlands, `barteld@philos.rug.nl`

PDL. In this paper we investigate the expressive power of epistemic logic with relativized common knowledge (EL-RC) and public announcement logic with common knowledge. Based on the results in [4, 3] we also show how reduction axioms can be acquired for epistemic PDL. These yield reduction axioms that can be used to can analyze how common knowledge is attained in some major communication types.

# 2 Logics of Public Announcement

## 2.1 Language and Semantics of PAL

Public announcement logic (PAL) was first developed by Plaza [11]. A public announcement is an epistemic update where all agents commonly know that they learn a certain formula. This is modeled by an operator $[\varphi]$, where $[\varphi]\psi$ is read as '$\psi$ holds after the announcement of $\varphi$'. The languages $\mathscr{L}_{\mathsf{PAL}}$ and $\mathscr{L}_{\mathsf{PAL\text{-}C}}$ (with common knowledge) are interpreted in epistemic models.

**Definition 1 (Epistemic models)** Let a finite set of propositional variables $P$ and a finite set of agents $N$ be given. An epistemic model is a triple $M = (W, R, V)$ such that

- $W \neq \emptyset$ is a set of possible worlds,
- $R : N \to \wp(W \times W)$ assigns an accessibility relation $R(a)$ to each agent $a$, and
- $V : P \to \wp(W)$ assigns a set of worlds to each propositional variable. $\square$

In epistemic logic the relations are usually restricted to equivalence relations. Here we treat the general modal case. The semantics are defined with respect to models with a distinguished 'actual world': $M, w$.

**Definition 2 (Semantics of PAL and PAL-C)** Let a model $M, w$ with $M = (W, R, V)$ be given. For atomic propositions, negations, conjunctions, we take the usual definition. Let $\varphi, \psi \in \mathscr{L}_{\mathsf{PAL}}$.

$$\begin{array}{lll} M, w \models \Box_a \varphi & \text{iff} & M, v \models \varphi \text{ for all } v \text{ such that } (w, v) \in R(a) \\ M, w \models [\varphi]\psi & \text{iff} & M, w \models \varphi \text{ implies } M|\varphi, w \models \psi \\ M, w \models C_B \varphi & \text{iff} & M, v \models \varphi \text{ for all } v \text{ such that } (w, v) \in R(B)^* \end{array}$$

where $R(B) = \bigcup_{a \in B} R(a)$, and $R(B)^*$ is its reflexive transitive closure. The updated model $M|\varphi = (W', R', V')$ is defined by restricting $M$ to $\varphi$-worlds. Let

$$[\![\varphi]\!] = \{v \in W | M, v \models \varphi\}.$$

Now let $W' = [\![\varphi]\!]$, $R'(a) = R(a) \cap [\![\varphi]\!]^2$, and $V'(p) = V(p) \cap [\![\varphi]\!]$. $\square$

A completeness proof for public announcement logic without an operator for common knowledge (PAL) is easy using reduction axioms.

**Definition 3 (Proof system for PAL)** The proof system for PAL is that for multi-modal S5 epistemic logic plus the following reduction axioms:

$$\begin{array}{lll} \textbf{At} & [\varphi]p \leftrightarrow (\varphi \to p) & \text{(atoms)} \\ \textbf{PF} & [\varphi]\neg\psi \leftrightarrow (\varphi \to \neg[\varphi]\psi) & \text{(partial functionality)} \\ \textbf{Dist} & [\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi) & \text{(distribution)} \\ \textbf{KA} & [\varphi]\Box_a\psi \leftrightarrow (\varphi \to \Box_a[\varphi]\psi) & \text{(knowledge-announcement)} \end{array}$$

as well as an inference rule of necessitation for all announcement modalities. $\square$

The formulas on the left of these equivalences are of the form $[\varphi]\psi$. In **At** the announcement operator no longer occurs on the right-hand side. In the other reduction axioms formulas within the scope of an announcement are of higher complexity on the left than on the right.

For public announcement logic including a common knowledge operator (PAL-C), a completeness proof with reduction axioms is impossible. There is no reduction axiom for formulas of the form $[\varphi]C_B\psi$, given the expressivity results in [2].

### 2.2 Relativized Common Knowledge: EL-RC

The semantic intuition for $[\varphi]C_B\psi$ is clear, however. Let a $B$-path be a sequence of worlds $w_0, \ldots, w_n$ such that for all $i < n$ there is an $a \in B$ such that $(w_i, w_{i+1}) \in R(B)$. We can now say when a formula of the form $[\varphi]C_B\psi$ is true as follows: if $\varphi$ is true in the old model, then every $B$-path in the new model ends in a $\psi$ world. This implies that in the old model every $B$-path that consists exclusively of $\varphi$-worlds ends in a $[\varphi]\psi$ world. To facilitate this, we use the operator $C_B(\varphi, \psi)$, which expresses that every $B$-path which consists exclusively of $\varphi$-worlds ends in a $\psi$ world. $C_B(\varphi, \psi)$ can be paraphrased as 'If $\varphi$ were announced to $B$, then it would be common knowledge among $B$ that $\psi$ was the case'. It is called *relativized common knowledge*.

**Definition 4 (Language and Semantics of EL-RC)** The language of EL-RC is that of EL, together with the operator for relativized common knowledge, with semantics given by:

$$M, w \models C_B(\varphi, \psi) \text{ iff } M, v \models \psi \text{ for all } v \text{ such that } (w, v) \in (R(B) \cap [\![\varphi]\!]^2)^*$$

where $(R(B) \cap [\![\varphi]\!]^2)^*$ is the reflexive transitive closure of $R(B) \cap [\![\varphi]\!]^2$. □

Note that common knowledge relativized to $\varphi$ is *not* what results from a public update with $\varphi$. E.g., $[p]C_B\Diamond_a\neg p$ is *not* equivalent to $C_B(p, \Diamond_a\neg p)$, for $[p]C_B\Diamond_a\neg p$ is always false, and $C_B(p, \Diamond_a\neg p)$ holds in models where every $p$ path ends in a world with an $a$ successor with $\neg p$. We will show that $C_B(p, \Diamond_a\neg p)$ cannot be expressed in PAL-C in Section 2.4. The semantics of the other operators is standard. Ordinary common knowledge can be defined with the new notion: $C_B\varphi \equiv C_B(\top, \varphi)$. To obtain a proof system for EL-RC we need just a slight adaptation of the usual axioms. To prove completeness for our extended static language, one can follow [9]. Details can be found in [8].

### 2.3 Reduction Axioms for PAL-RC

Next, let PAL-RC be the logic with both relativized common knowledge and public announcements. We can find a reduction axiom for $[\varphi]C_B(\psi, \chi)$, the formula that expresses that after public announcement of $\varphi$, every $\psi$ path leads to a $\chi$ world. Note that this holds exactly in those worlds where every $\varphi$ path where announcing $\varphi$ makes $\psi$ true ends in a world where announcing $\varphi$ makes $\chi$ true. This observation yields the following proof system for PAL-RC:

**Definition 5 (Proof system for PAL-RC)** The proof system for PAL-RC is that for EL-RC plus the reduction axioms for PAL, together with:

$$\textbf{C-Red } [\varphi]C_B(\psi, \chi) \leftrightarrow C_B(\varphi \wedge [\varphi]\psi, [\varphi]\chi) \text{ (common knowledge reduction)}$$

as well as an inference rule of necessitation for all announcement modalities. □

This proof system is sound and complete. It turns out that PAL-RC is no more expressive than EL-RC by a direct translation, where the translation clause for $[\varphi]C_B(\psi, \chi)$ relies on the above

$$
\begin{tikzcd}
\mathsf{PAL} \arrow[r] \arrow[d, leftrightarrow] & \mathsf{PAL\text{-}C} \arrow[r] & \mathsf{PAL\text{-}RC} \arrow[d, leftrightarrow] \\
\mathsf{EL} \arrow[r] & \mathsf{EL\text{-}C} \arrow[r] \arrow[u] & \mathsf{EL\text{-}RC}
\end{tikzcd}
$$

Figure 1: The expressive power of various epistemic logics. $\mathsf{L} \longrightarrow \mathsf{L}'$ means that $\mathsf{L}'$ is more expressive than $\mathsf{L}$.

insight. For details, see [8]. It also provides a precise characterization how common knowledge can be achieved. One open problem posed in [8] is whether EL-RC is more expressive than PAL-C.

### 2.4 Expressivity via Model Comparison Games for EL-RC

We provide characteristic games for the logics presented so far.

**Definition 6 (Model comparison games)** Let two epistemic models $M = (W, R, V)$ and $M' = (W', R', V')$ be given. Starting from each $w \in W$ and $w' \in W'$, the *n-round model comparison game* between Spoiler and Duplicator is given as follows. If $n = 0$ Spoiler wins if $w$ and $w'$ differ in their atomic properties, otherwise Duplicator wins. Otherwise Spoiler makes one of the scenarios available to him dependent on the operators in the logical language in each round:

$\Box_a$**-move** Spoiler chooses a point $x$ in one model which is an $a$-successor of the current $w$ or $w'$, and Duplicator responds with an $a$-successor $y$ of the other current world in the other model. The output is $x, y$.

$C_B$**-move** Spoiler chooses a point $x$ in one model which is reachable by a $B$-path from $w$ or $w'$, and Duplicator responds by choosing a matching world $y$ in the other model. The output is $x, y$.

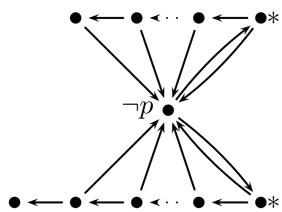
$RC_B$**-move** Spoiler chooses a $B$-path $x_0 \dots x_n$ in either of the models with $x_0$ the current $w$ or $w'$. Duplicator responds with a $B$-path $y_0 \dots y_m$ in the other model, with $y_0 = w'$. Then Spoiler can (a) make the end points $x_n, y_m$ the output of this round, or (b) he can choose a world $z$ on Duplicators path, and Duplicator must respond by choosing a matching world $u$ on Spoilers path, and $z, u$ is the output.

$[\varphi]$**-move** Spoiler chooses a number $r < n$, and sets $S \subseteq W$ and $S' \subseteq W'$, with the current $w \in S$ and likewise $w' \in S'$. *Stage 1*: Duplicator chooses states $s$ in $S \cup S'$, $\overline{s}$ in $\overline{S} \cup \overline{S'}$. Then Spoiler and Duplicator play the $r$-round game for these worlds. If Duplicator wins this subgame, she wins the $n$-round game. *Stage 2*: Otherwise, the game continues in the relativized models $M|S, w$ and $M'|S', w'$ over $n - r$ rounds.

After each move the game continues with the new output states. If these differ in their atomic properties, Spoiler wins – otherwise, a player loses whenever he cannot perform a move while it is his turn. If Spoiler has not won after all $n$ rounds, Duplicator wins the whole game. $\Box$

With these games we can show the expressivity results summarized in Figure 1. In [2] it was already shown that PAL-C is more expressive than EL-C.

We can show that EL-RC is more expressive than PAL-C with models of the type shown below.




The idea is that Duplicator cannot distinguish the top line from the bottom line of these models when they are long enough. The formula

$$C(p, \neg\Box p)$$

is true in the starred world in the top line, but false in the starred world in the bottom line. However there is not one formula in $\mathscr{L}_{\mathsf{PAL\text{-}C}}$ that can distinguish these world for all the models of the type shown above. The idea leads to the following theorem.

**Theorem 1** EL-RC is more expressive than PAL-C. $\Box$

# 3 A New Logic of Communication and Change

Our methodology for epistemic logic with announcements from Section 2 also works more generally. In this section, we make the same move in the general dynamic logic of updates with events involving combinations of communication and actual change, which also lacks a reduction axiom for common knowledge. This will provide a general method for finding reduction axioms for any update. The method be used to find particular reduction axiom for major communication types.

## 3.1 Update Models and their Execution

Dynamic *updates* with epistemic aspects, such as communication or other information-bearing events, are quite similar to static epistemic *situations*. In [2] this analogy is used as the engine for general update of epistemic models under epistemic actions. Here we extend this with substitutions that effect changes in valuations at particular worlds.

**Substitutions** $\mathscr{L}$ substitutions distribute over all language constructs, and map all but a finite number of basic propositions to themselves. $\mathscr{L}$ substitutions can be represented as sets of bindings $\{p_1 \mapsto \varphi_1, \ldots, p_n \mapsto \varphi_n\}$ where all the $p_i$ are different, and where no $\varphi_i$ is equal to $p_i$. If $\sigma$ is a $\mathscr{L}$ substitution, then the set $\{p \in P \mid \sigma(p) \neq p\}$ is called its *domain*, notation $\mathrm{dom}(\sigma)$. Use $\epsilon$ for the identity substitution. Let $\mathrm{SUB}_{\mathscr{L}}$ be the set of all $\mathscr{L}$ substitutions.

**Definition 7 (Epistemic Models under a Substitution)** If $M = (W, V, R)$ is an epistemic model and $\sigma$ is a $\mathscr{L}$ substitution (for an appropriate epistemic language $\mathscr{L}$), then $V_M^\sigma$ is the valuation given by $\lambda w \lambda p \cdot w \in [\![\sigma(p)]\!]^M$. In other words, $V_M^\sigma$ assigns to $w$ the set of basic propositions $p$ such that $\sigma(p)$ is true in world $w$ in model $M$. For $M = (W, V, R)$, call $M^\sigma$ the model given by $(W, V_M^\sigma, R)$. $\Box$

**Definition 8 (Update models)** An update model for a finite set of agents $N$ with a language $\mathscr{L}$ is a quadruple $\mathsf{U} = (\mathsf{E}, \mathsf{R}, \mathsf{pre}, \mathsf{sub})$ where $\mathsf{E} = \{\mathsf{e}_0, \dots, \mathsf{e}_{n-1}\}$ is a finite non-empty set of events, $\mathsf{R} : N \to \wp(\mathsf{E}^2)$ assigns an accessibility relation $\mathsf{R}(a)$ to each agent $a \in N$, $\mathsf{pre} : \mathsf{E} \to \mathscr{L}$ assigns a precondition to each event, $\mathsf{sub} : \mathsf{E} \to \mathrm{SUB}_{\mathscr{L}}$ assigns a $\mathscr{L}$ substitution to each event. A pair $\mathsf{U}, \mathsf{e}$ is an update model with a distinguished actual event $\mathsf{e} \in \mathsf{E}$. □

The effect of executing an update is modeled by the following *product construction*; our definition extends the definition of [2] by taking the effects of the substitutions into account.

**Definition 9 (Update Execution)** Given a static epistemic model $M = (W, R, V)$, a world $w \in W$, an action model $\mathsf{U} = (\mathsf{E}, \mathsf{R}, \mathsf{pre}, \mathsf{sub})$ and an action state $\mathsf{e} \in \mathsf{E}$ with $M, w \models \mathsf{pre}(\mathsf{e})$, we say that the result of executing $\mathsf{U}, \mathsf{e}$ in $M, w$ is the static model $M \circ \mathsf{U}, (w, \mathsf{e}) = (W', R', V'), (w, \mathsf{e})$ where $W' = \{(v, \mathsf{f}) \mid M, v \models \mathsf{pre}(\mathsf{f})\}$, $R'(a) = \{((v, \mathsf{f}), (u, \mathsf{g})) \mid (v, u) \in R(a) \text{ and } (\mathsf{f}, \mathsf{g}) \in \mathsf{R}(a)\}$, $V'(u, \mathsf{f}) = V_M^{\mathsf{sub}(\mathsf{f})}(u)$. □

Definitions 8 (with all substitutions equal to $\epsilon$) and 9 provide a semantics for the logic of epistemic actions $\mathsf{LEA}$ of [2]. The language $\mathscr{L}_{\mathsf{LEA}}$ contains formulas $[\mathsf{U}, \mathsf{e}]\varphi$, where a $\mathsf{U}$ is any *finite update model* for $\mathscr{L}_{\mathsf{LEA}}$. These say that 'every execution of $\mathsf{U}, \mathsf{e}$ yields a model where $\varphi$ holds'. In [2] a proof system is presented for $\mathsf{LEA}$ with a complicated completeness proof, and without reduction axioms for common knowledge. So, we will extend this language to get reduction axioms after all. Again, the semantic intuition about the crucial case $M, w \models [\mathsf{U}, \mathsf{e}]C_B\varphi$ is clear. It says that, if there is a $B$-path $w_0, \dots, w_n$ (with $w_0 = w$) in the static model and a matching $B$-path $\mathsf{e}_0, \dots, \mathsf{e}_n$ (with $\mathsf{e}_0 = \mathsf{e}$) in the update model with $M, w_i \models \mathsf{pre}(\mathsf{e}_i)$ for all $i \leq n$, then $M, w_n \models \varphi$. These are the sort of object that *propositional dynamic logic* ($\mathsf{PDL}$) is very well suited for reasoning about.

### 3.2 Epistemic PDL

We assume the reader is familiar with the language and semantics of $\mathsf{PDL}$. For a textbook introduction see [7]. In our epistemic perspective, relational atoms will be viewed as (the epistemic accessibilities of) single agents. Also, if $B \subseteq N$ and $B$ is finite, use $B$ as shorthand for $b_1 \cup b_2 \cup \cdots$. Under this convention, the general knowledge operator $E_B\varphi$ takes the shape $[B]\varphi$, while the common knowledge operator $C_B\varphi$ appears as $[B^*]\varphi$, i.e., $[B]\varphi$ expresses that it is general knowledge among agents $B$ that $\varphi$, and $[B^*]\varphi$ expresses that it is common knowledge among agents $B$ that $\varphi$. In the special case where $B = \emptyset$, $B$

### 3.3 LCC, a Dynamic Logic of Communication and Change

Now we have all the ingredients for the definition of the logic of communication and change that allows reduction axioms.

**Definition 10 (LCC, Language)** The language $\mathscr{L}_{\mathsf{LCC}}$ is the result of adding a clause $[\mathsf{U}, \mathsf{e}]\varphi$ for update execution to the language of $\mathsf{PDL}$, where $\mathsf{U}$ is an update model for $\mathscr{L}_{\mathsf{LCC}}$. □

**Definition 11 (LCC, Semantics)** The semantics is the standard semantics of $\mathsf{PDL}$, with the meaning of $[\mathsf{U}, \mathsf{e}]\varphi$ in $M = (W, R, V)$ given by:

$$M, w \models [\mathsf{U}, \mathsf{e}]\varphi \text{ iff } M, w \models \mathsf{pre}(\mathsf{e}) \text{ implies } M \circ \mathsf{U}, (w, \mathsf{e}) \models \varphi.$$

□

### 3.4 Expressive Power: Reducing $\mathsf{LCC}$ to Epistemic $\mathsf{PDL}$

In order to reduce $\mathsf{LCC}$ to $\mathsf{PDL}$ we need a reduction axiom for formulas of the form $[\mathsf{U},\mathsf{e}][\pi]\varphi$. As before, the quest for reduction axioms starts with the attempt to describe what is the case after the update in terms of what is the case before the update. In the case of $\mathsf{LCC}$, epistemic relations can take the shape of arbitrary $\mathsf{PDL}$ programs. So we must ask ourselves how we can find, for a given relation $[\![\pi]\!]^{M\circ\mathsf{U}}$ a corresponding relation in the original model $M,w$.

A formula of the form $\langle\mathsf{U},\mathsf{e}_i\rangle\langle\pi\rangle\varphi$ is true in some model $M,w$ iff there is a $\pi$-path in $M\circ\mathsf{U}$ leading from $(w,\mathsf{e}_i)$ to a $\varphi$ world $(v,\mathsf{e}_j)$. That means there is some path $w\ldots v$ in $M$ and some path $\mathsf{e}_i\ldots\mathsf{e}_j$ in $\mathsf{U}$ such that $(M,w)\models\mathsf{pre}(\mathsf{e}_i)$ and $\ldots$ and $(M,v)\models\mathsf{pre}(\mathsf{e}_j)$ and of course $(M,v)\models\langle\mathsf{U},\mathsf{e}_j\rangle\varphi$. The program $T^{\mathsf{U}}_{ij}(\pi)$ captures this. A $T^{\mathsf{U}}_{ij}(\pi)$-path in the original model corresponds to a $\pi$-path in the updated model. But in defining $T^{\mathsf{U}}_{ij}(\pi)$ we cannot refer to a model $M$. The definition of $T^{\mathsf{U}}_{ij}(\pi)$ only depends on $\pi$, $\mathsf{U}$, $\mathsf{e}_i$ and $\mathsf{e}_j$. These transformers are used in the reduction axiom, which can be formulated as follows:

$$[\mathsf{U},\mathsf{e}_i][\pi]\varphi \quad\leftrightarrow\quad \textstyle\bigwedge_{j=0}^{n-1}[T^{\mathsf{U}}_{ij}(\pi)][\mathsf{U},\mathsf{e}_j]\varphi.$$

The program transformer $T^{\mathsf{U}}_{ij}$ is defined as follows:

**Definition 12 ($T^{\mathsf{U}}_{ij}$ Program Transformers)**

$$\begin{array}{lcllcl}
T^{\mathsf{U}}_{ij}(\pi_1;\pi_2) &=& \bigcup_{k=0}^{n-1}(T^{\mathsf{U}}_{ik}(\pi_1);T^{\mathsf{U}}_{kj}(\pi_2)) & T^{\mathsf{U}}_{ij}(a) &=& \begin{cases} ?\mathsf{pre}(\mathsf{e}_i);a & \text{if } \mathsf{e}_iR(a)\mathsf{e}_j,\\ ?\bot & \text{otherwise}\end{cases}\\
T^{\mathsf{U}}_{ij}(\pi_1\cup\pi_2) &=& T^{\mathsf{U}}_{ij}(\pi_1)\cup T^{\mathsf{U}}_{ij}(\pi_2) & & & \\
T^{\mathsf{U}}_{ij}(\pi^*) &=& K^{\mathsf{U}}_{ijn}(\pi) & T^{\mathsf{U}}_{ij}(?\varphi) &=& \begin{cases} ?(\mathsf{pre}(\mathsf{e}_i)\wedge[\mathsf{U},\mathsf{e}_i]\varphi) & \text{if } i=j,\\ ?\bot & \text{otherwise}\end{cases}
\end{array}$$

where $K^{\mathsf{U}}_{ijn}(\pi)$ is given by Definition 13. □

We need the program transformer $K^{\mathsf{U}}_{ijn}$ in order to build the paths corresponding to the transitive closure of $\pi$ in the updated model step by step, where we take more and more events into account.

**Definition 13 ($K^{\mathsf{U}}_{ijk}$ Path Transformers)** $K^{\mathsf{U}}_{ijk}(\pi)$ is defined by recursing on $k$, as follows:

$$K^{\mathsf{U}}_{ij0}(\pi) = \begin{cases} ?\top\cup T^{\mathsf{U}}_{ij}(\pi) & \text{if } i=j,\\ T^{\mathsf{U}}_{ij}(\pi) & \text{otherwise}\end{cases}$$

$$K^{\mathsf{U}}_{ij(k+1)}(\pi) = \begin{cases} (K^{\mathsf{U}}_{kkk}(\pi))^* & \text{if } i=k=j,\\ (K^{\mathsf{U}}_{kkk}(\pi))^*;K^{\mathsf{U}}_{kjk}(\pi) & \text{if } i=k\neq j,\\ K^{\mathsf{U}}_{ikk}(\pi);(K^{\mathsf{U}}_{kkk}(\pi))^* & \text{if } i\neq k=j,\\ K^{\mathsf{U}}_{ijk}(\pi)\cup(K^{\mathsf{U}}_{ikk}(\pi);(K^{\mathsf{U}}_{kkk}(\pi))^*;K^{\mathsf{U}}_{kjk}(\pi)) & \text{otherwise } (i\neq k\neq j).\end{cases}$$

□

**Theorem 2 (Reduction)** Assume $\mathsf{U}$ has $n$ states $\mathsf{e}_0,\ldots,\mathsf{e}_{n-1}$. Then:

$$M,w\models[\mathsf{U},\mathsf{e}_i][\pi]\varphi \text{ iff } M,w\models\bigwedge_{j=0}^{n-1}[T^{\mathsf{U}}_{ij}(\pi)][\mathsf{U},\mathsf{e}_j]\varphi.$$

What the Reduction Theorem gives us is that $\mathsf{LCC}$ is equivalent to $\mathsf{PDL}$, and that a proof system for $\mathsf{LCC}$ can be given in terms of axioms that reduce formulas of the form $[\mathsf{U},\mathsf{e}]\varphi$ to equivalent formulas $\psi$. This approach makes clear that it is possible to view the updates as a kind of finite automata. This was used in [8] to obtain reduction axioms for automata $\mathsf{PDL}$ with update models. These results point the way to appropriate reduction axioms for $\mathsf{LCC}$, as follows.

**Definition 14 (Proof system for LCC)** The proof system for LCC consists of all axioms and rules of PDL plus the following reduction axioms:

$$\begin{array}{lll}
[\mathsf{U},\mathsf{e}]p & \leftrightarrow & (\mathsf{pre}(\mathsf{e}) \to p^{\mathsf{sub}(\mathsf{e})}) \\
[\mathsf{U},\mathsf{e}]\neg\varphi & \leftrightarrow & (\mathsf{pre}(\mathsf{e}) \to \neg[\mathsf{U},\mathsf{e}]\varphi) \\
[\mathsf{U},\mathsf{e}](\varphi_1 \wedge \varphi_2) & \leftrightarrow & ([\mathsf{U},\mathsf{e}]\varphi_1 \wedge [\mathsf{U},\mathsf{e}]\varphi_2) \\
[\mathsf{U},\mathsf{e}_i][\pi]\varphi & \leftrightarrow & \bigwedge_{j=0}^{n-1}[T_{ij}^{\mathsf{U}}(\pi)][\mathsf{U},\mathsf{e}_j]\varphi.
\end{array}$$

and necessitation for update model modalities. □

Thus, we see that LCC is no more expressive than PDL; indeed, we can translate LCC to PDL.

**Theorem 3 (Completeness for LCC)** $\models \varphi$ iff $\vdash \varphi$. □

**Proof** The proof system for PDL is complete and every formula in $\mathscr{L}_{\mathsf{LCC}}$ is provably equivalent to one in $\mathscr{L}_{\mathsf{PDL}}$. □

# 4 Analyzing Major Communication Types

The program transformation approach provides a systematic perspective on communicative updates. In the case of public announcement and common knowledge, it was still possible to generate appropriate reduction axioms by hand. Such axioms can also be generated automatically by program transformation.

The update model for a public announcement that $\varphi$ consists of a single state $\mathsf{e}_0$ with precondition $\varphi$ and the universal relation for all agents. Call this model $P_\varphi$. By program transformations we find the following reduction axiom

$$[P_\varphi,\mathsf{e}_0][B^*]\psi \quad\leftrightarrow\quad [(?\varphi;B)^*][P_\varphi,\mathsf{e}_0]\psi.$$

Compare this to the special purpose operator $C_B(\varphi,\psi)$ from Section 2.2.

Also consider a secret group communication $\mathrm{CC}_\varphi^B,\mathsf{e}_0$ where $\varphi$ is sent to group $B$, and the other agents think nothing happens ($\mathrm{CC}_\varphi^B,\mathsf{e}_1$)), the program transformation for common belief among group $D$ (which may or may not overlap with $B$) works out as follows:

$$[\mathrm{CC}_\varphi^B,\mathsf{e}_0][D^*]\psi \quad\leftrightarrow\quad [(?\varphi;(B\cap D))^*][\mathrm{CC}_\varphi^B,\mathsf{e}_0]\psi \wedge [(?\varphi;(B\cap D))^*;(D\setminus B);D^*][\mathrm{CC}_\varphi^B,\mathsf{e}_1]\psi.$$

Compare [12] for a direct axiomatization of the logic of CCs.

Finally, we consider group messages. This example is one of the simplest cases that shows that program transformations gives us reduction axioms that are no longer feasible to give by hand. The update model for a group message to $B$ that $\varphi$ consists of two states $\mathsf{e}_0,\mathsf{e}_1$, where $\mathsf{e}_0$ has precondition $\varphi$ and $\mathsf{e}_1$ has precondition $\top$, and where the accessibilities $T$ are given by:

$$T = \{\mathsf{e}_0 R(b)\mathsf{e}_0 \mid b\in B\} \cup \{\mathsf{e}_1 R(b)\mathsf{e}_1 \mid b\in B\} \cup \{\mathsf{e}_0 R(a)\mathsf{e}_1 \mid a\in N\setminus B\} \cup \{\mathsf{e}_1 R(a)\mathsf{e}_0 \mid a\in N\setminus B\}.$$

Abbreviating $D\cup(D\setminus B;(?\varphi;D)^*;?\varphi;D\setminus B)$ as $\pi$, we get the following transformation for common knowledge among $D$ after a group message to $B$ that $\varphi$:

$$\begin{array}{lll}
[G_\varphi^B,\mathsf{e}_0][D^*]\psi & \leftrightarrow & [(?\varphi;D)^* \cup ((?\varphi;D)^*;?\varphi;D\setminus B;\pi^*;D\setminus B;(?\varphi;D)^*)][G_\varphi^B,\mathsf{e}_0]\psi \wedge \\
 & & [(?\varphi;D)^*;?\varphi;D\setminus B;\pi^*][G_\varphi^B,\mathsf{e}_1]\psi.
\end{array}$$

# 5 Conclusion

Update logics provide excellent means for studying exchange of factual and higher-order information. In this many-agent setting, common knowledge is an essential concept. We have taken two extended languages for update logic that admit explicit update/common knowledge reduction axioms: one (EL-RC) for public announcement only, and one (PDL) for general update. They allow us to investigate many communication types of which we have given some examples. The program transformations give us detailed analyses of the change in common knowledge due to communicative updates.

**Acknowledgements** The authors would like to thank Tomasz Sadzik for his comment on an extended version of this paper.

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