Formal Concept Analysis and Prototypes

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1 Introduction

Categorization is probably one of the most central areas in the study of cognition, language and information. However, there is a serious gap running through the semantic treatments of categories and concepts [3]. On one side we find the 'classical', formal approach, based on logical considerations, that has lent itself well for computational applications. In this approach, concepts are defined in terms of necessary and sufficient conditions. On the other side is an informal approach to categorization that is usually motivated by the results of psychological experiments and that has not found its way into technologies on a large scale. Concepts here are based on prototypes, stereotypical attributes and family resemblances, which have become the hallmark of cognitive semantics. Obviously, it is important to bridge this gap, for theoretical and practical reasons.

We explore Formal Concept Analysis (FCA) [1, Ch 3], [2] as a way to do this. The conclusion of our preliminary investigation will be that a lattice theoretical approach to concepts is a suitable starting point for investigating and formalizing notions like prototypicality and family resemblance and for making them more relevant for ontology design and other applications.

2 Formal Concept Analysis

Contexts in FCA are triples (O, A, H) consisting of a domain O of *objects*, a domain A of *attributes* and a relation H on $O \times A$, with $(o, a) \in H$ expressing that o has a. FCA represents a concept in a given context (O, A, H) as a pair (X, Y) with $X \subseteq O$ and $Y \subseteq A$ satisfying the two conditions:

$$X = \{x \in O \mid \forall y \in Y \ xHy\}, \quad Y = \{y \in A \mid \forall x \in X \ xHy\}.$$

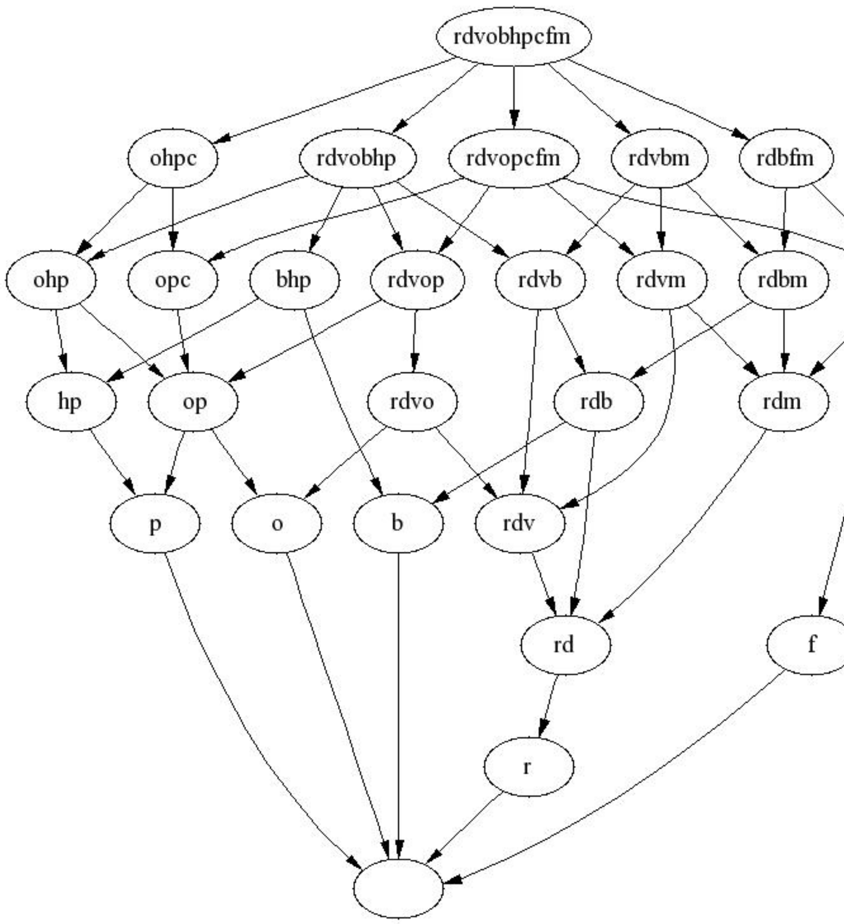
If (X, Y) is a concept, X is called its extent, Y its intent.

Using X' for the set of those attributes that are shared by all members of X and Y' for the set of all objects that share the attributes in Y, we can say (as is well known) that (X, Y) is a concept iff $X = Y'$ and $Y = X'$ iff $X = X''$ and $Y = X'$ iff $Y = Y''$ and $X = Y'$.

Our running example will be the following zoological example context with ten objects and nine attributes.

	walks	quacks	lays eggs	feathered	warmblooded	flies	sings	small	suckles
robin			√	√	√	√	√	√	
dove			√	√	√	√		√	
vulture			√	√	√	√			
ostrich	√		√	√	√				
bat					√	√		√	√
horse	√				√				√
platypus	√		√		√				√
crocodile	√		√						
frog		√	√					√	
mosquito			√			√		√	

Here are the concepts of this context:



The concept lattice that is determined by sets O and A, and relation $H \subseteq O \times A$ is the set $(P, \leq)$, where

$$P = \{(X, Y) \mid X \subseteq O, Y \subseteq A, (X, Y) \text{ is a concept } \},$$

and $\leq$ is given by:

$$(X_1, Y_1) \leq (X_2, Y_2) :\equiv X_1 \subseteq X_2.$$

The picture above represents the 'covers' relation that goes with $\leq$, where y covers x ($y \rhd x$) or x is covered by y ($x \lhd y$), if $x < y$ and $x \leq z < y$ implies $x = y$.

3 Extended Concepts

FCA can be used to analyze the *structural* properties of conceptual domains that give rise to prototypical categories. The lattice-based analysis reveals natural discontinuities and groupings in contexts and it can show why certain objects are more representative (prototypical) of a concept than others or why certain attributes have a higher *cue validity* for a concept, following the seminal psychological work of Rosch [4]. One of the key insights of the work of Rosch is that prototypical members of a category have more attributes in common with other members of that category and less with members of other categories and the members of cluster concepts (like 'furniture') have more attributes in common with each other than with other objects. FCA makes it possible to relate such properties to the partial ordering and connectivity of a concept lattice, as we will show.

Standard FCA can help us determine structural factors in a context that underlie prototypicality, but it cannot explicitly *represent* the prototype structure in a concept. We propose an extended notion of concept that allows for the representation of prototypical objects and stereotypical attributes. Prototypical objects are objects that somehow exemplify a concept to a greater extent than non-prototypical objects. Similarly, stereotypical attributes are attributes that are somehow more basic than non-stereotypical attributes.

First we define an extended context as a context that in addition singles out a subset from the set of attributes as *essential* attributes. Formally, an extended context is a quadruple (O, A, H, E) such that (O, A, H) is a context, and $E \subseteq A$.

Let an extended context (O, A, H, E) be given. Let (X, P, Y, Q) be a quadruple with $P \subseteq X \subseteq O$ and $Y \subseteq Q \subseteq A$. We say that (X, P, Y, Q) is an extended concept in (O, A, H, E) when (X, Y) and (P, Q) are both concepts in (O, A, H), and moreover $Y \subseteq E$ (all attributes that every object in the concept has are essential attributes). P are the prototypical instances of the concept and $Q - Y$ are its stereotypical attributes. Note that it follows immediately from this definition that $(P, Q) \leq (X, Y)$, in other words, that (X, Y) is a super-concept of (P, Q).

Extended concepts again form a complete lattice. Meets are now given by

$$(\bigcap_{i \in I} X_i, \bigcap_{i \in I} P_i, \left(\bigcup_{i \in I} Y_i\right)'', \left(\bigcup_{i \in I} Q_i\right)''),$$

joins by

$$(\left(\bigcup_{i\in I} X_i\right)'', \left(\bigcup_{i\in I} P_i\right)'', \bigcap_{i\in I} Y_i, \bigcap_{i\in I} Q_i).$$

Thus, the present formalization would predict that the prototypical cases of the concept that results from combining (X_1, P_1, Y_1, Q_1) and (X_2, P_2, Y_2, Q_2) are the objects in $P_1 \cap P_2$.

We can define a pre-order relation $\preceq$ for 'being closer to the prototypical case' on the extent of an extended concept (X, P, Y, Q), as follows:

$$x_1 \preceq x_2 :\equiv \forall y \in (Q - Y)(x_2 H y \to x_1 H y).$$

Intuitively, $x_1 \preceq x_2$ expresses that x_1 has all the stereotypical attributes of x_2. The poset reflection of $\preceq$ then yields equivalence classes in the extent of the extended concept that give concepts a graded prototype structuring.

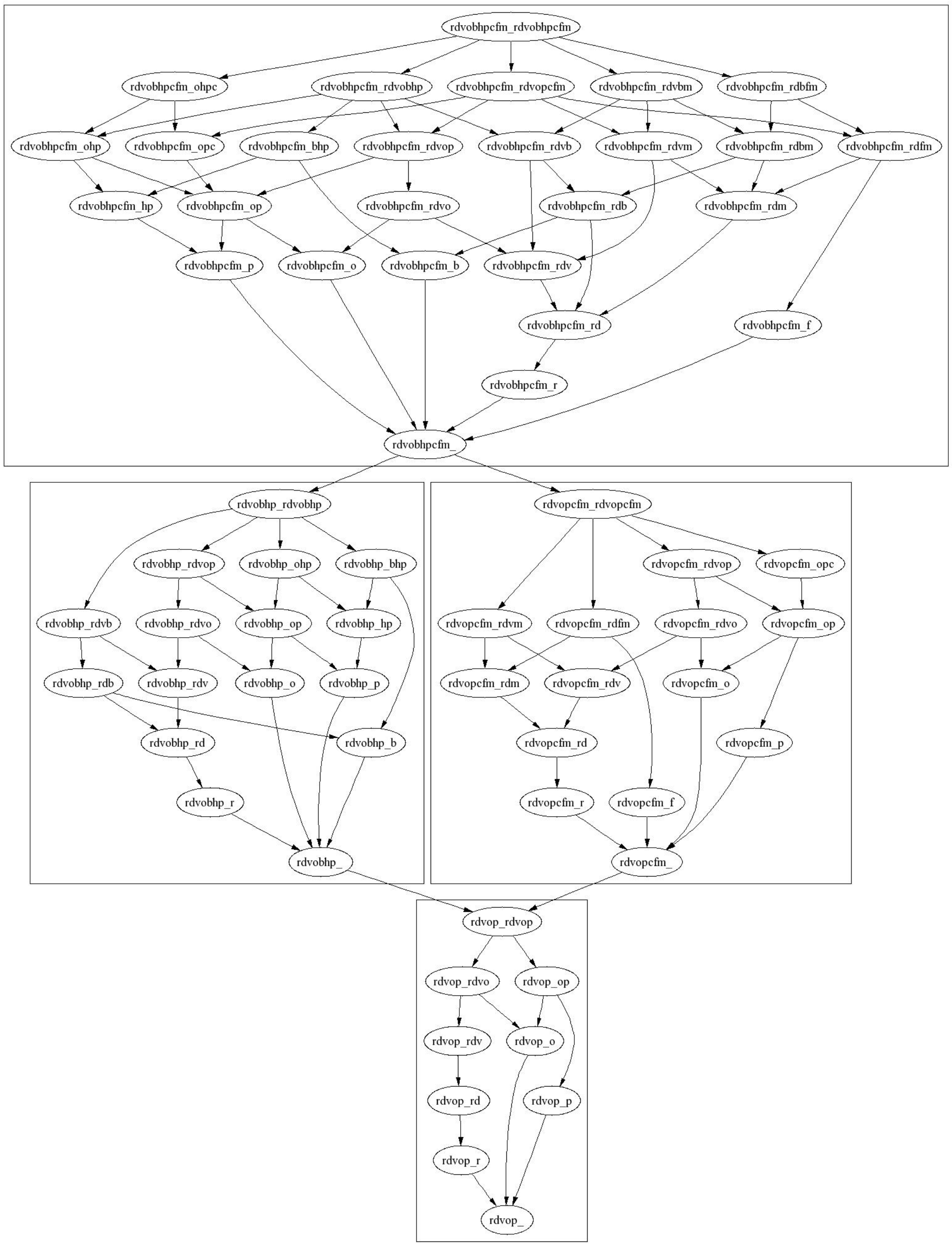
The notion of a concept in formal concept analysis is completely extensional. If (X_1, Y_1) and (X_2, Y_2) are concepts with $X_1 = X_2$ then the concepts are the same. A concept is fully determined by its extent. For extended concepts this is no longer so. If (X_1, P_1, Y_1, Q_1) and (X_2, P_2, Y_2, Q_2) are extended concepts and their extents are the same, their sets of prototypes may still vary.

To make the zoological example context into an extended context, we must specify its essential attributes. E.g., we can focus on birds by singling out *egg-laying* and *warmblooded* as essential. Singling out *warmblooded*, and *suckling* as essential would provide an appropriate way for focusing on mammals.

One of the (many) non-trivial extended concepts for this extended context is:

({robin,dove,vulture,ostrich,bat,horse,platypus}, {robin,dove},
{ warmblood}, {eggs,feathers,warmblood,flies,small})

This extended concept puts the birds and the mammals together (through the attribute *warmblooded*), and it singles out the small feathered flying egg layers (robin and dove) as prototypes. Although this is not yet completely accurate (psychologically, or linguistically), it illustrates the role that extended concepts can play in modeling prototype structure in concepts.



The picture above gives the lattice of extended concepts for this example, with concepts satisfying none of the essential attributes grouped together at the top, concepts satisfying the essential attribute *warmblooded* middle left, concepts satisfying the essential attribute *egg-laying* middle right, and concepts sharing the two essential attributes at the bottom. Notice that the special status of the duckbilled platypus (p) as a warmblooded egglayer becomes quite clear in the structure of the extended context.

4 Conclusions

Although Formal Concepts in the sense of FCA by themselves are not rich enough to model prototypicality, FCA still is an interesting point of departure for a formal study of prototypicality and for connecting model-theoretic methods, cognitive semantic data and practical applications. It would be interesting to extend the account to lexical negation and complex concepts. Also, connections between different contexts should be explored, and finally, the whole apparatus should be set to work in the analysis of specific lexical and ontological domains.

References

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