Jankov's Theorems for Intermediate Logics in the Setting of Universal Models

Dick de Jongh
Institute for Logic, Language and Computation, University of Amsterdam
Fan Yang
Department of Mathematics and Statistics, University of Helsinki

Abstract

In this article we prove the Jankov Theorem for extensions of **IPC** ([6]) and the Jankov Theorem for **KC** ([7]) in a uniform frame-theoretic way in the setting of n-universal models for **IPC**. In frame-theoretic terms, the first Jankov Theorem states that for each finite rooted frame there is a formula ψ with the property that any counter-model for ψ needs this frame in the sense that each descriptive frame that falsifies ψ will have this frame as the p-morphic image of a generated subframe. The second one states that **KC** is the strongest logic that proves no negationless formulas beyond **IPC**. On the way we give a simple proof of the fact discussed and proved in [1] that the upper part of the n-Henkin model $\mathcal{H}(n)$ is isomorphic to the n-universal model $\mathcal{U}(n)$ of **IPC**. All these results earlier occurred in a somewhat different form in [8].

1 Introduction

In this article we prove the Jankov Theorem for extensions of intuitionistic logic **IPC** ([6]) and the Jankov Theorem for Jankov's logic **KC** ([7]) in a uniform frame-theoretic way in the setting of n-universal models for **IPC**. In frame-theoretic terms, the first Jankov Theorem states that for each finite rooted frame there is a formula ψ with the property that any counter-model for ψ needs this frame in the sense that each descriptive frame that falsifies ψ will have this frame as the p-morphic image of a generated subframe. The second one states that **KC** is the strongest logic that proves no negationless formulas beyond **IPC**.

The first Jankov theorem is proved in Section 3, the second one in Section 4. Section 2 introduces n-universal models and n-Henkin models and develops their relationship sufficiently for the proofs in Sections 3 and 4. In section 5 we conclude our very straightforward proof that the upper part of the n-Henkin model $\mathcal{H}(n)$ is isomorphic to the n-universal model $\mathcal{U}(n)$ of **IPC**. This theorem was discussed extensively and proved in [1] in a more algebraic manner.

We will use the standard Kripke frames ($\mathfrak{F} = \langle W, R\rangle$), descriptive frames ($\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$) and models ($\mathfrak{M} = \langle W, R, V\rangle$) for intuitionistic propositional logic **IPC**, including the notation $\mathfrak{M}, w \vDash \varphi$. We extend the notation $V(p)$ to formulas: $V(\varphi) = \{w \in W \mid w \vdash \varphi\}$. Our models will usually be n-models, i.e. models with the valuation V restricted to the atoms $p_1, \ldots, p_n$ and thereby to n-formulas, formulas formed from $p_1, \ldots, p_n$. If X is a set of elements in the frame $\mathfrak{F}$ we will write $\mathfrak{F}_X$ for the subframe of $\mathfrak{F}$ generated by X, shortening this to $\mathfrak{F}_w$ if X is a single element w; similarly for models. We call the upward closed subsets of W (with respect to the relation R) *upsets*. The set of all upsets of W is denoted by $Up(W)$.

We have the usual notions of p-morphism for Kripke frames, descriptive frames and models.

Definition 1.1.

1. Let $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle W', R' \rangle$ be two Kripke frames. A map f from W to V is called a *(Kripke frame) p-morphism* of $\mathfrak{F}$ to $\mathfrak{G}$ if it satisfies the following conditions:
   - For any $w, u \in W$, wRu implies $f(w)R'f(u)$;
   - $f(w)R'v'$ implies $\exists v \in W(wRv \wedge f(v) = v')$.
2. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $\mathfrak{G} = \langle W', R', \mathcal{P}' \rangle$ be two descriptive frames. We call a Kripke frame p-morphism f of $\langle W, R \rangle$ to $\langle W', R' \rangle$ a (descriptive frame) p-morphism of $\mathfrak{F}$ onto $\mathfrak{G}$, if it also satisfies the following condition:
   - $\forall X \in \mathcal{P}'$, $f^{-1}(X) \in \mathcal{P}$.
3. A Kripke frame p-morphism f of $\mathfrak{F}$ to $\mathfrak{G}$ is called a *p-morphism* of a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ to a model $\mathfrak{N} = \langle \mathfrak{G}, V' \rangle$ if
   - $w \in V(p) \Longleftrightarrow f(w) \in V'(p)$ for every $p \in$ PROP.

The canonical n-model resulting from the usual completeness proof will be called the *n-Henkin model*. It consists of n-theories with the disjunction property. Our first business will be the development of the n-universal models and their relationship to the n-Henkin model.

2 n-universal models and n-Henkin models of IPC

In this section we recall the definition of an n-universal model. Throughout this section, we will talk about the valuation of point w in a n-model $\mathfrak{M}$ by using the term *color*. In general, an *n-color* is a 0-1-sequence $c_1 \cdots c_n$ of length n. If the length is understandable from the context, we will use *color* instead of n-color. The set of all n-colors is denoted by $\mathbf{C}^n$.

We define an ordering on the colors as follows:

$$c_1 \cdots c_n \leq c'_1 \cdots c'_n \text{ iff } c_i \leq c'_i \text{ for each } 1 \leq i \leq n.$$

We write $c_1 \cdots c_n < c'_1 \cdots c'_n$ if $c_1 \cdots c_n \leq c'_1 \cdots c'_n$ but $c_1 \cdots c_n \neq c'_1 \cdots c'_n$.

A *coloring* on a nonempty set W is a function $col : W \to \mathbf{C}^n$. Colorings on frames $\langle W, R\rangle$ will have to satisfy $uRv \Rightarrow col(u) \leq col(v)$. Then colorings and valuations on frames are in one-one correspondence. Given a $\mathfrak{M} = \langle W, R, V\rangle$, we can describe the valuation of a point by the coloring $col_V : W \to \mathbf{C}^n$, defined by $col_V(w) = c_1 \cdots c_n$, where for each $1 \leq i \leq n$,

$$c_i = \begin{cases} 1, & w \in V(p_i); \\ 0, & w \notin V(p_i). \end{cases}$$

We call $col_V(w)$ *the color of* w *under* V.

In any frame $\mathfrak{F} = \langle W, R\rangle$, we say that a subset $X \subseteq W$ *totally covers* a point $w \in W$, denoted by $w \prec X$, if X is the set of all immediate successors of w. We will just write $w \prec v$ in the case that $w \prec \{v\}$. A subset $X \subseteq W$ is called an *anti-chain* if $|X| > 1$ and for every $w, v \in X$, $w \neq v$ implies that $\neg wRv$ and $\neg vRw$.

We can now inductively define the n*-universal model* $\mathcal{U}(n)$ by its cumulative layers $\mathcal{U}(n)^k$ for $k \in \omega$.

Definition 2.1.

- The first layer $\mathcal{U}(n)^1$ consists of 2^n nodes with the 2^n different n-colors under the discrete ordering.
- Under each element w in $\mathcal{U}(n)^k - \mathcal{U}(n)^{k-1}$, for each color $s < col(w)$, we put a new node v in $\mathcal{U}(n)^{k+1}$ such that $v \prec w$ with $col(v) = s$, and we take the reflexive transitive closure of the ordering.
- Under any finite anti-chain X with at least one element in $\mathcal{U}(n)^k - \mathcal{U}(n)^{k-1}$ and any color s with $s \leq col(w)$ for all $w \in X$, we put a new element v in $\mathcal{U}(n)^{k+1}$ such that $col(v) = s$ and $v \prec X$ and we take the reflexive transitive closure of the ordering.

The whole model $\mathcal{U}(n)$ is the union of its layers. It is easy to see from the construction that every $\mathcal{U}(n)^k$ is finite. As a consequence, the generated submodel $\mathcal{U}(n)_w$ is finite for any node w in $\mathcal{U}(n)$.

The 1-universal model is also called *Rieger-Nishimura ladder*, which is depicted in Figure 1.

Let $Upper(\mathfrak{M})$ denote the submodel $\mathfrak{M}_{\{w \in W | d(w) < \omega\}}$ generated by all the points with finite depth. It is known that the n-universal model is isomorphic to the finite part of the n-Henkin model $Upper(\mathcal{H}(n))$. N. Bezhanishvili gave in [1] an algebraic proof of this fact. In the final section, we prove it directly on the basis of two important lemmas that we already need in the next section on the first Jankov theorem. These two lemmas respectively state that every finite model can be mapped p-morphically onto a generated submodel of $\mathcal{U}(n)$ (Lemma 2.2), and that $\mathcal{U}(n)_w$ is isomorphic to the submodel of $\mathcal{H}(n)$ generated by the theory of the de Jongh formula of w (Lemma 2.9, see Definition 2.4).

Theorem 2.2. *For any finite rooted Kripke* n*-model* $\mathfrak{M}$*, there exists a unique point* w *in* $\mathcal{U}(n)$ *and a p-morphism of* $\mathfrak{M}$ *onto* $\mathcal{U}(n)_w$.

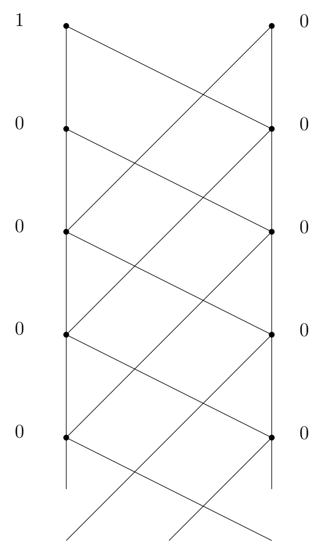

Figure 1: Rieger-Nishimura ladder

For a proof of Theorem 2.2, see e.g. [8]. It implies immediately, by the finite model property of **IPC**, that $\mathcal{U}(n)$ is a counter-model to every formula that is a non-theorem of **IPC**. This shows that $\mathcal{U}(n)$ deserves being called a “universal model”.

Theorem 2.3.

1. *For any* n*-formula* $\varphi, \mathcal{U}(n) \models \varphi$ *iff* $\vdash_{\mathbf{IPC}} \varphi$*.*

2. *For any* n*-formulas* φ, ψ*, for all* $w \in \mathcal{U}(n)(w \models \varphi \Rightarrow w \models \psi)$ *iff* $\varphi \vdash_{\mathbf{IPC}} \psi$*.*

Proof. (1) $\Leftarrow$: trivial. $\Rightarrow$: Suppose $\nvdash_{\mathbf{IPC}} \varphi$. Then there exists a finite n-model $\mathfrak{M}$ and a point $w \in \mathfrak{M}$ such that $\mathfrak{M}, w \not\models \varphi$. By Theorem 2.2, there exists a p-morphism f of $\mathfrak{M}$ to $\mathcal{U}(n)$. Hence, $\mathcal{U}(n), f(w) \not\models \varphi$.

(2) follows easily from (1). □

For any node w in an n-model $\mathfrak{M}$, if $\{w_1, ..., w_m\}$ is the set of all immediate successors of w, then we let

$prop(w) := \{p_i \,|\, w \models p_i, 1 \leq i \leq n\}$,
$notprop(w) := \{q_i, |\, w \not\models q_i, 1 \leq i \leq n\}$,
$newprop(w) := \{r_j, |\, w \not\models r_j$ and $w_i \models r_j$ for each $1 \leq i \leq m,$ for $1 \leq j \leq n\}$[1].

Here $newprop(w)$ denotes the set of atoms which are about to be true in w, i.e. they are true in all of w's proper successors. Next, we define the formulas φ_w and ψ_w, which were first introduced in [4], and which were extensively discussed and named de Jongh formulas in [1].

[1] Note that if w is an endpoint, $newprop(w) = notprop(w)$.

Definition 2.4. Let w be a point in $\mathcal{U}(n)$. We inductively define its *de Jongh formulas* φ_w and ψ_w.

If $d(w) = 1$, then let

$$\varphi_w := \bigwedge prop(w) \wedge \bigwedge \{\neg p_k \,|\, p_k \in notprop(w),\ 1 \leq k \leq n\},$$
$$\psi_w := \neg\varphi_w.$$

If $d(w) > 1$, and $\{w_1, ..., w_m\}$ is the set of all immediate successors of w, then define

$$\varphi_w := \bigwedge prop(w) \wedge (\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi_{w_i} \to \bigvee_{i=1}^{m} \varphi_{w_i}),$$
$$\psi_w := \varphi_w \to \bigvee_{i=1}^{m} \varphi_{w_i}.$$

The most important properties of the de Jongh formulas are revealed in the next theorem. It was first proved in [4].

Theorem 2.5. *For every $w \in \mathcal{U}(n) = \langle U(n), R, V\rangle$, we have that*

- $V(\varphi_w) = R(w)$,
  where $R(w) = \{u \in \mathcal{U}(n) \,|\, wRu\}$*;*
- $V(\psi_w) = U(n) \setminus R^{-1}(w)$,
  where $R^{-1}(w) = \{u \in \mathcal{U}(n) \,|\, uRw\}$.

An easy lemma that is needed in the proof of Jankov's theorem in the next section is the following.

Lemma 2.6. *If $u, v \in \mathcal{U}(n)$ and vRu, then* $\vdash_{\mathbf{IPC}} \varphi_u \to \varphi_v$.

Proof. Immediate from Theorem 2.5 and Theorem 2.3. □

Definition 2.7. We write

- $Cn(\varphi) = \{\psi \,|\, \vdash_{\mathbf{IPC}} \varphi \to \psi\}$.
  We write $Cn_n(\varphi) = \{n\text{-formula } \psi \,|\, \vdash_{\mathbf{IPC}} \varphi \to \psi\}$, but may leave the n out if it is clear from the context.
- $Th(\mathfrak{M}, w) = \{\varphi \,|\, \mathfrak{M}, w \models \varphi\}$.
  We write $Th(w) = \{\varphi \,|\, w \models \varphi\}$ if $\mathfrak{M}$ is clear from the context, and $Th_n(\mathfrak{M}, w)$ for the restriction of $Th(\mathfrak{M}, w)$ to the set of n-formulas. Again, we may delete the n.

Corollary 2.8. *For any point w in $\mathcal{U}(n)$, $Th(w) = Cn_n(\varphi_w)$.*

Proof. By Theorem 2.5, $Th(w) \supseteq Cn_n(\varphi_w)$. For the other direction, let ψ be an n-formula such that $\mathcal{U}(n), w \models \psi$. By Theorem 2.5 again, we have that $\mathcal{U}(n) \models \varphi_w \to \psi$, thus by Theorem 2.3, $\vdash_{\mathbf{IPC}} \varphi_w \to \psi$, i.e. $\psi \in Cn_n(\varphi_w)$. □

The next lemma expresses the essence of the fact that the upper part of the n-Henkin model is isomorphic to the n-universal model. We will pursue this in the last section. For the time being the lemma will come in very useful in the proof of the first Jankov Theorem, the main theorem of the next section.

Lemma 2.9. *For any $w \in \mathcal{U}(n)$, let φ_w be a de Jongh formula. Then we have that $\mathcal{H}(n)_{Cn(\varphi_w)} \cong \mathcal{U}(n)_w$.*

Proof. Let $\mathcal{U}(n) = \langle U(n), R, V\rangle$ and $\mathcal{H}(n) = \langle H(n), R', V'\rangle$. Define a map $f : U(n)_w \to \mathcal{H}(n)_{Cn(\varphi_w)}$ by taking

$$f(v) = Cn(\varphi_v).$$

We show that f is an isomorphism.

First for any $v \in U(n)$, by Corollary 2.8, we have that $v \in V(p)$ iff $Cn(\varphi_v) \in V'(p)$ and that

$$\begin{aligned} uRv &\Longleftrightarrow \mathcal{U}(n), v \models \varphi_u \text{ (by Theorem 2.5)} \\ &\Longleftrightarrow \varphi_u \in Cn(\varphi_v) \text{ (by Corollary 2.8)} \\ &\Longleftrightarrow Cn(\varphi_u) \subseteq Cn(\varphi_v) \\ &\Longleftrightarrow f(u)R'f(v). \end{aligned}$$

This makes f into a homomorphism.

Now, suppose $u \neq v$; w.l.o.g. we may assume that $\neg uRv$, which by Theorem 2.5 means that $\mathcal{U}(n), u \not\models \varphi_v$. Thus, $\varphi_v \notin Cn(\varphi_u)$ by Corollary 2.8, and so $f(u) = Cn(\varphi_u) \neq Cn(\varphi_v) = f(v)$. Hence, f is injective.

It remains to show that f is surjective. That is, to show that for any $\Gamma \in \mathcal{H}(n)_{Cn(\varphi_u)}$ (i.e. any n-theory $\Gamma \supseteq Cn_n(\varphi_u)$ with the disjunction property) there exists v with uRv such that $\Gamma = Cn(\varphi_v)$. We prove this by induction on the depth of u.

$d(u) = 1$. It suffices to show that if $Cn(\varphi_u) \subseteq \Gamma$, then $\Gamma = Cn(\varphi_u)$. This is clear from the fact that $\theta \in Cn(\varphi_u)$ iff $\vdash_{\mathbf{IPC}} \varphi_u \to \theta$ iff (by Corollary 2.8), because this shows that $Cn(\varphi_u)$ is maximal consistent.

$d(u) = k+1$. Let $\{u_1, ..., u_m\}$ be the set of all immediate successors of u. Suppose $Cn(\varphi_u) \subseteq \Gamma$. If $Cn(\varphi_{u_i}) \subseteq \Gamma$ for some $1 \leq i \leq m$, then by induction hypothesis, $\Gamma = Cn(\varphi_v)$ for some $v \in R(u_i)$, i.e. $v \in R(u)$. So, we can assume $Cn(\varphi_{u_i}) \nsubseteq \Gamma$ for all $1 \leq i \leq m$. Thus $\Gamma \nvdash \varphi_{u_i}$ for each $1 \leq i \leq m$. Take any $\theta \in \Gamma$. We then have also $\theta \wedge \varphi_u \in \Gamma$. So,

$$\theta \wedge \varphi_u \nvdash \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}.$$

Since $\mathcal{U}(n)$ is universal, there exists a $u' \in U(n)$ such that

$$\mathcal{U}(n), u' \models \theta \wedge \varphi_u \text{ and } \mathcal{U}(n), u' \not\models \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}.$$

By Theorem 2.5, $u' = u$, which implies that $\mathcal{U}(n), u \models \theta$. By Corollary 2.8, $\theta \in Cn(\varphi_u)$. Therefore $\Gamma = Cn(\varphi_u)$. ☐

We end this section by a corollary which follows from the correspondence between $\mathcal{H}(n)$ and $\mathcal{U}(n)$, and which plays a crucial role in our proof of Jankov's theorem.

Corollary 2.10. *Let $\mathfrak{M}$ be any model and w be a point in $\mathcal{U}(n) = \langle W, R, V \rangle$. For any point x in $\mathfrak{M}$, if $\mathfrak{M}, x \models \varphi_w$, then there exists a unique point v satisfying*

$$\mathfrak{M}, x \models \varphi_v,\ \mathfrak{M}, x \not\models \varphi_{v_1}, \cdots, \mathfrak{M}, x \not\models \varphi_{v_m},$$

where $v \prec \{v_1, \cdots, v_m\}$, and wRv.

Proof. Note that $Th_n(\mathfrak{M}, x)$ is a node in $\mathcal{H}(n) = \langle W', R', V' \rangle$. $\mathfrak{M}, x \models \varphi_w$ implies that $Th_n(\mathfrak{M}, x) \vdash_{\mathbf{IPC}} \varphi_w$ and $Cn_n(\varphi_w) R' Th_n(\mathfrak{M}, x)$. Thus, by Lemma 2.9, $Th_n(\mathfrak{M}, x) = Cn_n(\varphi_v)$ for a unique point $v \in W$. Moreover we have wRv. So $\mathfrak{M}, x \models \varphi_v$.

By Theorem 2.5, we have that $\mathcal{U}(n) \not\models \varphi_v \to \varphi_{v_i}$ for all $1 \leq i \leq m$. Thus $\nvdash_{\mathbf{IPC}} \varphi_v \to \varphi_{v_i}$ and $\varphi_{v_i} \notin Cn_n(\varphi_v) = Th_n(\mathfrak{M}, x)$, so $\mathfrak{M}, x \not\models \varphi_{v_i}$. □

3 Jankov's Theorem for extensions of IPC

The original theorem was proved by Jankov in [6] with respect to algebraically inspired formulas. De Jongh proved in [4] the same theorem with regard to the de Jongh formulas defined above. Here we transform the latter proof, which made an algebraic detour, into a purely frame-theoretic one. We have set the stage in the previous section in such a manner that the analogies between the proof of the Jankov theorem and the proof of our central Lemma 4.7 for the Jankov Theorem on **KC** (Theorem 4.9) in the next section will come out as clearly as possible.

One of the things we will need in the proof of Jankov's theorem is that under certain conditions a Kripke frame p-morphism from a descriptive frame to a finite descriptive frame is almost automatically also a descriptive frame p-morphism. The next lemma states the necessary conditions.

Lemma 3.1. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $\mathfrak{G} = \langle W', R', \mathcal{P}' \rangle$ be two descriptive frames with W' finite. Let f be a (Kripke frame) p-morphism from the Kripke frame $\langle W, R \rangle$ to the Kripke frame $\langle W', R' \rangle$ such that $f^{-1}(R(w))$ is an admissible set for any $w \in W'$. Then f is also a (descriptive frame) p-morphism from the descriptive frame $\mathfrak{F}$ to the descriptive frame $\mathfrak{G}$.*

Proof. It suffices to show that for any $X \in \mathcal{P}'$, $f^{-1}(X) \in \mathcal{P}$. Observing that $X = \bigcup\limits_{w \in X} R(w)$, we obtain that

$$f^{-1}(X) = f^{-1}(\bigcup_{w \in X} R(w)) = \bigcup_{w \in X} f^{-1}(R(w)),$$

which implies $f^{-1}(X) \in \mathcal{P}$ since $f^{-1}(X)$ is a finite union of admissible sets. □

The following useful lemma was introduced (as Theorem 3.2.16) and discussed in [1]. It says that any finite rooted frame can be isomorphically found as a generated submodel of $\mathcal{U}(n)$ if only we take n large enough.

Lemma 3.2. *For any finite rooted frame $\mathfrak{F} = \langle W', R' \rangle$, there exists a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ on $\mathfrak{F}$ such that $\mathfrak{M}$ is isomorphic to a generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$.*

Proof. We introduce a propositional variable p_w for every point w in W, and define a valuation V by letting $V(p_w) = R(w)$. Put $n = |W|$. By Theorem 2.2, there exists a p-morphism f from the model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ onto a generated submodel $\mathcal{U}(n)_w$. By the construction, we know that different points of $\mathfrak{M}$ have different colors, thus f is injective, i.e. $\mathfrak{M}$ is isomorphic to $\mathcal{U}(n)_w$. $\square$

Note that the underlying Kripke frame of $\mathcal{U}(n)_w = \langle W, R, V \rangle$ described in the previous lemma can be viewed as the general frame $\langle W, R, Up(W) \rangle$, which is a descriptive frame since W is finite.

Theorem 3.3 (Jankov). *For every finite rooted frame $\mathfrak{F}$, let ψ_w be the de Jongh formula of w in the model $\mathcal{U}(n)_w$ described in Lemma 3.2. Then for every descriptive frame $\mathfrak{G}$,*

$\mathfrak{G} \not\models \psi_w$ iff $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$.

Proof. The direction from right to left is obvious, since $\mathfrak{F} \not\models \psi_w$ follows immediately from Theorem 2.5.

For the other direction, suppose $\mathfrak{G} \not\models \psi_w$. Then there exists a model $\mathfrak{N}$ on $\mathfrak{G}$ such that

$$\mathfrak{N} \not\models \varphi_w \to \varphi_{w_1} \vee \cdots \vee \varphi_{w_m}, \tag{1}$$

where $w \prec \{w_1, \cdots, w_m\}$. Consider the generated submodel $\mathfrak{N}' = \mathfrak{N}_{V'(\varphi_w)} = \langle W', R', \mathcal{P}', V' \rangle$ of $\mathfrak{N}$. Note that since $V'(\varphi_w)$ is admissible, $\langle W', R', \mathcal{P}' \rangle$ is a descriptive frame. Define a map $f : W' \to W$ by taking $f(x) = v$ iff

$$\mathfrak{N}', x \models \varphi_v,\ \mathfrak{N}', x \not\models \varphi_{v_1}, \cdots, \mathfrak{N}', x \not\models \varphi_{v_k}, \tag{2}$$

where $v \prec \{v_1, \cdots, v_k\}$.

Note that for every $x \in W'$, $\mathfrak{N}', x \models \varphi_w$, thus by Corollary 2.10, there exists a unique $v \in R(w)$ satisfying (2). So f is well-defined.

We show that f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Suppose $x, y \in W'$ with $xR'y$, $f(x) = v$ and $f(y) = u$. Since $\mathfrak{N}', x \models \varphi_v$, we have that $\mathfrak{N}', y \models \varphi_v$. By Corollary 2.10, there exists a unique point $u' \in W$ such that u' and y satisfy (2), moreover, vRu'. So, since u and y also satisfy (2), by the uniqueness, $u' = u$ and vRu.

Next, suppose $x \in W'$ and $v, u \in W$ such that $f(x) = v$ and vRu. We now show that there exists $y \in W'$ with $xR'y$ such that

$$\mathfrak{N}', y \models \varphi_u,\ \mathfrak{N}', y \not\models \varphi_{u_1}, \cdots, \mathfrak{N}', y \not\models \varphi_{u_l} \tag{3}$$

where $u \prec \{u_1, \cdots, u_l\}$. This will give us the required $f(y) = u$. We will prove this directly if u is an immediate successor of v, i.e. one of the v_i. For u in general it follows then by tracing a chain from v to u.

Since x and v satisfy (2), and φ_v implies by its definition that

$$\bigvee_{i=1}^{m} \psi_{v_i} \to \bigvee_{i=1}^{k} \varphi_{v_i}, \tag{4}$$

we must have that

$$\mathfrak{N}', x \not\models \psi_u, \tag{5}$$

because u is one of the v_i. From (5) the existence of y with $xR'y$ satisfying (3) immediately follows. Hence, we have shown that f is a (Kripke frame) p-morphism.

To show that f is surjective it is sufficient to note that, by (1), there exists $x \in W'$ such that (2) holds for x and w, i.e. $f(x) = w$. Then, for every node $v \in W$, we have that wRv. Since f is a (Kripke frame) p-morphism, there exists $y \in R'(x) \subseteq W'$ such that $f(y) = v$.

It remains to show that f is a (descriptive frame) p-morphism between the two descriptive frames. In view of Lemma 3.1, it is sufficient to show that for any $v \in X$, $f^{-1}(R(v)) = V'(\varphi_v)$ which is an admissible set.

Indeed, for every $x \in f^{-1}(R(v))$, there exists $u \in R(v)$ such that $f(x) = u$ and so $\mathfrak{N}', x \models \varphi_u$. Applying Lemma 2.6 gives $\mathfrak{N}', x \models \varphi_v$, and so $x \in V'(\varphi_v)$. On the other hand, for every $x \in V'(\varphi_v)$, by Corollary 2.10, there exists a unique $u \in R(v)$ such that $f(x) = u$, thus $x \in f^{-1}(R(v))$.

Hence f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Then since $\mathfrak{F} \cong \langle W, R, \mathcal{P} \rangle$, $\mathfrak{F}$ is a p-morphic image of $\langle W', R', \mathcal{P}' \rangle$, which is a generated subframe of $\mathfrak{G}$. □

We conclude this section with a useful theorem of [4], [5]. We will not apply it directly in this paper, but we will use an adapted form of it in the special case of the next section.

Theorem 3.4. *If L is an intermediate logic strictly extending* **IPC***, i.e.* **IPC** $\subset L \subseteq$ **CPC***, then there exists $n \in \omega$ and w in $\mathcal{U}(n)$ such that $L \vdash \psi_w$.*

Proof. Suppose χ is a formula satisfying

$$L \vdash \chi \text{ and } \mathbf{IPC} \nvdash \chi.$$

Then there exists a finite rooted frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \chi$. By Lemma 3.2, there exists a model $\langle \mathfrak{F}, V \rangle$ on $\mathfrak{F}$ such that $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$ for some generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$. Consider the de Jongh formula ψ_w. Suppose $L \nvdash \psi_w$. Then there exists a descriptive frame $\mathfrak{G}$ of L such that $\mathfrak{G} \not\models \psi_w$. By Theorem 3.3, $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. Thus, $\mathfrak{F}$ is an L frame. Since $L \vdash \chi$, we have that $\mathfrak{F} \models \chi$, which gives us a contradiction. □

4 Jankov's Theorem for KC

Jankov's logic **KC** (also called the logic of weak decidability) is the intermediate logic axiomatized by $\neg\varphi \vee \neg\neg\varphi$. **KC** is complete with respect to finite rooted frames with unique top points. From that fact it is not difficult to show that **KC** proves exactly the same negation-free formulas as **IPC**. That is, for any negation-free formula φ, **KC** $\vdash \varphi$ iff **IPC** $\vdash \varphi$. For all this, check for example [3]. Jankov proved in [7] that **KC** is the strongest intermediate logic that has this property. Another proof can be obtained by using canonical formulas (see [3]). In this section, we give a frame-theoretic alternative proof of Jankov's Theorem. The basic idea of the proof comes from adapting the proof of Theorem 3.3 combined with Theorem 3.4 to the special case of **KC**-frames.

We start with defining formulas φ'_w and ψ'_w, which are negation-free modifications of de Jongh formulas. They function on **KC**-frames as de Jongh formulas do on all frames. First, we introduce some terminology.

For any finite set X of formulas with $|X| > 1$, let

$$\Delta X = \bigwedge\{\varphi \leftrightarrow \psi \mid \varphi, \psi \in X\}.$$

For the case that $|X| = 1$ or 0, we stipulate $\Delta X = \top$.

Let $\mathcal{U}(n)_{w_0} = \langle W, R, V\rangle$ be a generated submodel with a largest element t of $\mathcal{U}(n)$ such that

- $t \models p_1 \wedge \cdots \wedge p_n$;
- $col(w) \neq col(v)$ for all $w, v \in W$ such that $w \neq v$.

Let r be a new propositional variable (to be identified with p_{n+1} so that we can talk about $p_1, \ldots, p_n, r$-models as $n+1$-models).

Definition 4.1. We inductively define the formulas φ'_w and ψ'_w for every $w \in W$.

If $d(w) = 1$,

$$\varphi'_w = p_1 \wedge \cdots \wedge p_n,$$
$$\psi'_w = \varphi'_w \to r.$$

If $d(w) = 2$, let q be an arbitrary propositional letter in $notprop(w)$. Define

$$\varphi'_w = \bigwedge prop(w) \wedge \Delta notprop(w) \wedge ((q \to r) \to q)^2,$$
$$\psi'_w = \varphi'_w \to q.$$

If $d(w) > 2$ and $w \prec \{w_1, \cdots, w_m\}$, then let

$$\varphi'_w := \bigwedge prop(w) \wedge (\bigvee newprop(w) \vee \bigvee_{i=1}^{m} \psi'_{w_i} \to \bigvee_{i=1}^{m} \varphi'_{w_i}),$$

$$\psi'_w := \varphi'_w \to \bigvee_{i=1}^{m} \varphi'_{w_i}.$$

We will prove for the φ'_w and ψ'_w formulas a lemma (Lemma 4.7) which is analogous to Theorem 3.3 for the φ_w and ψ_w formulas. It is good to note already that the φ'_w and ψ'_w formulas cannot be evaluated in $\mathcal{U}(n)$, since there is one propositional variable to many in them. Nevertheless, we will be able to follow the general line of the argument in the previous section.

It is worth remarking that, for $d(w) = 2$, ψ'_w is a generalized form of Peirce's Law $(((q \to r) \to q) \to q)$.

Lemma 4.2. *$\varphi'_w[r/\bot]$[3] and $\psi'_w[r/\bot]$ are* **IPC**-*equivalent to φ_w and ψ_w, respectively.*

[2]Note that in the definition, it does not matter which $q \in notprop(w)$ is chosen. Note also that $notprop(w) = newprop(w)$.

[3]We write $\varphi[p/\psi]$ for the formula obtained by replacing all occurrences of p in φ by ψ.

Proof. We prove this by induction on $d(w)$.

$d(w) = 1$. Trivial.

$d(w) = 2$. $\varphi'_w[r/\bot] = \bigwedge prop(w) \wedge \Delta notprop(w) \wedge ((q \to \bot) \to q)$.

First note that $(q \to \bot) \to q)$ is equivalent to $\neg\neg q$. On the other hand,

$$\vdash \varphi_w \leftrightarrow \bigwedge prop(w) \wedge (\bigvee notprop(w) \vee \neg(p_1 \wedge \cdots \wedge p_n) \to p_1 \wedge \cdots \wedge p_n)$$
$$\vdash \varphi_w \leftrightarrow \bigwedge prop(w) \wedge (\bigvee notprop(w) \to p_1 \wedge \cdots \wedge p_n) \wedge (\neg(p_1 \wedge \cdots \wedge p_n) \to p_1 \wedge \cdots \wedge p_n)$$

Under the assumption $\bigwedge prop(w)$, $\bigvee notprop(w) \to p_1 \wedge \cdots \wedge p_n$ is equivalent to $\Delta notprop(w)$. Furthermore, $\neg(p_1 \wedge \cdots \wedge p_n) \to p_1 \wedge \cdots \wedge p_n$ is equivalent to $\neg\neg(p_1 \wedge \cdots \wedge p_n)$ and hence to to $\neg\neg p_1 \wedge \cdots \wedge \neg\neg p_n$. This, in its turn is under the assumptions $\bigwedge prop(w)$ and $\Delta notprop(w)$ equivalent to $\neg\neg q$. So, indeed, $\vdash \varphi_w \leftrightarrow \varphi'_w[r/\bot]$ and

$$\begin{aligned}
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi'_w[r/\bot] \to q) \\
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi'_w[r/\bot] \to p_1 \wedge \cdots \wedge p_n) \\
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi_w \to p_1 \wedge \cdots \wedge p_n) \\
&\vdash \psi'_w[r/\bot] \leftrightarrow (\varphi_w \to \varphi_t) \\
&\vdash \psi'_w[r/\bot] \leftrightarrow \psi_w.
\end{aligned}$$

$d(w) > 2$. This is proved easily by applying the induction hypothesis. ☐

Obviously, we could have defined φ'_w and ψ'_w in such a way that this lemma would have been a complete triviality, but we preferred giving a more intuitive definition.

We will use the following corollary later in the proof of Theorem 4.9.

Corollary 4.3. *For any generated submodel $\mathcal{U}(n)_{w_0}$ of $\mathcal{U}(n)$ as described above, any point w in $\mathcal{U}(n)_{w_0}$, $\nvdash_{\mathbf{IPC}} \psi'_w$.*

Proof. By Theorem 2.5, $\mathcal{U}(n)_{w_0} \not\models \psi_w$, thus, by the Lemma 4.2, the underlying frame of $\mathcal{U}(n)_{w_0}$ falsifies ψ'_w. Hence $\nvdash_{\mathbf{IPC}} \psi'_w$. ☐

The next lemma shows that the φ'_w formulas have the same property that using Theorem 2.3 was easy to prove for the φ_w formulas in Lemma 2.6. Note however that this theorem is not applicable to the φ'_w formulas. Here we prove the corresponding theorem directly from the construction of the φ'_w and ψ'_w formulas by a method that could have been applied to the φ_w formulas, but would have been unnecessarily complicated in that case.

Lemma 4.4. *Let $\mathcal{U}(n)_{w_0} = \langle W, R, V \rangle$ be a model as described above and let w, v be two nodes in W with wRv. Then we have that $\vdash_{\mathbf{IPC}} \varphi'_v \to \varphi'_w$.*

Proof. We prove the lemma by induction on $d(v)$.

If $d(v) = 1$, then $\varphi'_v = p_1 \wedge \cdots \wedge p_n$. Since wRv, we have that $prop(w) \subseteq \{p_1, \cdots, p_n\}$ and

$$\vdash \varphi'_v \to \bigwedge prop(w). \tag{6}$$

We show that $\vdash \varphi'_v \to \varphi'_w$ by induction on $d(w)$.

$d(w) = d(v) + 1 = 2$. Then for any $p, q \in notprop(w) \subseteq \{p_1, \cdots, p_n\}$ we have that

$$\vdash p_1 \wedge \cdots \wedge p_n \to (p \leftrightarrow q) \text{ and } \vdash p_1 \wedge \cdots \wedge p_n \to ((q \to r) \to q).$$

It follows that

$$\vdash \varphi'_v \to \Delta notprop(w) \text{ and } \vdash \varphi'_v \to ((q \to r) \to q).$$

Together with (6), we obtain

$$\vdash \varphi'_v \to \bigwedge prop(w) \wedge \Delta notprop(w) \wedge ((q \to r) \to q)$$

i.e. $\vdash \varphi'_v \to \varphi'_w$.

$d(w) > 2$. Let $w \prec \{w_1, \cdots, w_k\}$. Tthen for any immediate successor w_i of w, since $d(w_i) < d(w)$ by induction hypothesis, we have that $\vdash \varphi'_v \to \varphi'_{w_i}$, which implies that $\vdash \varphi'_v \to \bigvee_{i=1}^{k} \varphi'_{w_i}$ and that

$$\vdash \varphi'_v \to (\bigvee newprop(w) \vee \bigvee_{i=1}^{k} \psi'_{w_i} \to \bigvee_{i=1}^{k} \varphi'_{w_i}). \tag{7}$$

Together with (6), we obtain

$$\vdash \varphi'_v \to \bigwedge prop(w) \wedge (\bigvee newprop(w) \vee \bigvee_{i=1}^{k} \psi'_{w_i} \to \bigvee_{i=1}^{k} \varphi'_{w_i}) \tag{8}$$

i.e. $\vdash \varphi'_v \to \varphi'_w$.

If $d(v) = 2$, then since $prop(w) \subseteq prop(v)$, clearly (6) holds. We show $\vdash \varphi'_v \to \varphi'_w$ by induction on $d(w)$.

$d(w) = d(v) + 1$. Then $v = w_i$ and $\varphi'_v = \varphi'_{w_i}$ for some immediate successor w_i of w, hence $\vdash \varphi'_v \to \bigvee_{i=1}^{k} \varphi'_{w_i}$ and (7) follows. Together with (6), we obtain (8) i.e. $\vdash \varphi'_v \to \varphi'_w$.

$d(w) > d(v) + 1$. For any immediate successor w_i of w, since $d(w_i) < d(w)$, by the induction hypothesis, we have that $\vdash \varphi'_v \to \bigvee_{i=1}^{k} \varphi'_{w_i}$, which implies (7). Together with (6), we obtain (8) i.e. $\vdash \varphi'_v \to \varphi'_w$.

If $d(v) > 2$, then clearly $prop(w) \subseteq prop(v)$ gives (6). By a similar argument as above, we can show that (7) holds, thus, (8) i.e. $\vdash \varphi'_v \to \varphi'_w$ is obtained. □

Next, we want to prove for the φ'_w formulas an analogue to Corollary 2.10. But we will have to do this in two steps. First, we show that φ'_w nodes have the right color.

Lemma 4.5. *Let $\mathfrak{M} = \langle W', R', V' \rangle$ be any $n+1$-model and $\mathcal{U}(n)_{w_0} = \langle W, R, V \rangle$ be a model as described above. Put $V_n = V' \restriction \{p_1, \cdots, p_n\}$. For any point w in $\mathcal{U}(n)_{w_0}$ and any point x in $\mathfrak{M}$, if*

$$\mathfrak{M}, x \models \varphi'_w,\ \mathfrak{M}, x \not\models \varphi'_{w_1}, \cdots, \mathfrak{M}, x \not\models \varphi'_{w_m}, \tag{9}$$

where $w \prec \{w_1, \cdots, w_m\}$, then $col_{V_n}(x) = col_V(w)$.

Proof. We prove the lemma by induction on $d(w)$. In the following discussion we restrict attention to n-formulas and n-atoms all the time.

$d(w) = 1$, i.e. $w = t$. Then (9) means that $\mathfrak{M}, x \models p_1 \wedge \cdots \wedge p_n$. Also, $\mathcal{U}(n)_{w_0}, t \models p_1 \wedge \cdots \wedge p_n$. So $col_{V_n}(x) = col_V(w)$.

$d(w) = 2$. Then (9) implies that

$$\mathfrak{M}, x \models \bigwedge prop(w). \tag{10}$$

This means that all atoms true in w are true in x. From (9) we also have that

$$\mathfrak{M}, x \models \Delta notprop(w). \tag{11}$$

So, either all atoms false in w are false in x, or all are true in x. But, in this case, in (9) $m = 1$ and $w_1 = t$, so

$$\mathfrak{M}, x \not\models p_1 \wedge \cdots \wedge p_n. \tag{12}$$

This implies that all atoms false in w are false in x: $col_{V_n}(x) = col_V(w)$.

$d(w) > 2$. This is the induction step. Again we have as in the previous case that all atoms true in w are true in x. Now (9)

$$\mathfrak{M}, x \not\models \psi'_{w_i}, \tag{13}$$

for all immediate successor w_i of w, i.e. for each immediate successor w_i of w, there exists $y_i \in R'(x)$ such that y_i and w_i satisfy (9). Since $d(w_i) < d(w)$, by the induction hypothesis, we have that $col_{V_n}(y_i) = col_V(w_i)$. So, all atoms false in at least one of the w_i are false in x. On the other hand, (9) also implies

$$\mathfrak{M}, x \not\models \bigvee newprop(w), \tag{14}$$

So, all atoms true in all w_i but not in w are also false in x. We have $col_{V_n}(x) = col_V(w)$. □

This is the point where it becomes clear why at the start of this section we insisted on all the nodes of $\mathcal{U}(n)_{w_0}$ to have distinct colors. With this assumption the required analogue of (Corollary 2.10) now readily follows.

Lemma 4.6. *Let $\mathfrak{M}$ and $\mathcal{U}(n)_{w_0}$ be models described above. For any node w in $\mathcal{U}(n)_{w_0}$ and any node x in $\mathfrak{M}$, if $\mathfrak{M}, x \models \varphi'_w$, then there exists a unique point $v \in \mathcal{U}(n)_{w_0}$ satisfying*

$$\mathfrak{M}, x \models \varphi'_v,\ \mathfrak{M}, x \not\models \varphi'_{v_1}, \cdots, \mathfrak{M}, x \not\models \varphi'_{v_m}, \tag{15}$$

where $v \prec \{v_1, \cdots, v_m\}$, and wRv.

Proof. Suppose $\mathfrak{M}, x \models \varphi'_w$. We show that there exists $v \in R(w)$ satisfying (15) by induction on $d(w)$.

$d(w) = 1$. Then trivially $v = w$ satisfies (15).

$d(w) > 1$. If for all immediate successor w_i of w, $\mathfrak{M}, x \not\models \varphi'_{w_i}$, then w satisfies (15). Now suppose that for some immediate successor w_{i_0} of w, $\mathfrak{M}, x \models \varphi'_{w_{i_0}}$. Since $\mathfrak{M}, x \models \varphi'_{w_{i_0}}$ and $d(w_{i_0}) < d(w)$, by the induction hypothesis, there exists $v \in W$, such that $w_{i_0} R v$ and v satisfies (15). And clearly, wRv.

Next, suppose $v' \in \mathcal{U}(n)_{w_0}$ also satisfies (15). By Lemma 4.5,

$$col_V(v') = col_{V_n}(x) = col_V(v),$$

which by the property of $\mathcal{U}(n)_{w_0}$ means that $v' = v$. □

Let $\mathfrak{F}$ be a finite rooted frame with a largest element x_0. By Lemma 3.2, there exists a model $\langle \mathfrak{F}, V \rangle$ on $\mathfrak{F}$ such that $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$ for some generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$. Note that $\mathcal{U}(n)_w$ has a top point t, $t \models p_1 \wedge \cdots \wedge p_n$, and distinct points of $\mathcal{U}(n)_w$ have distinct colors.

The next lemma is a modification of the Jankov-de Jongh Theorem (Theorem 3.3) proved in the previous section. Both the statement of the lemma and its proof are generalized from those of Theorem 3.3.

Lemma 4.7. *For every finite rooted frame $\mathfrak{F}$ with a largest element, let $\mathcal{U}(n)_w$ be the model described above. Then for every descriptive frame $\mathfrak{G}$,*

$$\mathfrak{G} \not\models \psi'_w \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Proof. $\Leftarrow$: Let $\mathcal{U}(n)_w = \langle W, R, \mathcal{P}, V \rangle$. Suppose $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. By Theorem 2.5, $\mathcal{U}(n)_w \not\models \psi_w$, thus $\mathfrak{F} \not\models \psi_w$. By Lemma 4.2, we know in that case that $\mathfrak{F} \not\models \psi'_w$. Then $\mathfrak{G} \not\models \psi'_w$ follows immediately.

$\Rightarrow$: Suppose $\mathfrak{G} \not\models \psi'_w$. Then there exists a model $\mathfrak{N}$ on $\mathfrak{G}$ such that $\mathfrak{N} \not\models \psi'_w$. Consider the generated submodel $\mathfrak{N}' = \mathfrak{N}_{V'(\varphi'_w)} = \langle W', R', \mathcal{P}', V' \rangle$ of $\mathfrak{N}$. Since $V'(\varphi'_w)$ is admissible, $\mathfrak{N}'$ is descriptive. Define a map $f : W' \to W$ by taking $f(x) = v$ iff

$$\mathfrak{N}', x \models \varphi'_v,\ \mathfrak{N}', x \not\models \varphi'_{v_1}, \cdots, \mathfrak{N}', x \not\models \varphi'_{v_k}, \tag{16}$$

where $v \prec \{v_1, \cdots, v_k\}$.

Note that for every $x \in \mathfrak{N}'$, $\mathfrak{N}', x \models \varphi'_w$, thus by Lemma 4.6, there exists a unique $v \in R(w)$ satisfying (16). So f is well-defined.

We show that f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Suppose $x, y \in \mathfrak{N}'$ with $xR'y$, $f(x) = v$ and $f(y) = u$. Since $\mathfrak{N}', x \models \varphi'_v$, we have that $\mathfrak{N}', y \models \varphi'_v$. By Lemma 4.6, there exists a unique point $u' \in W$ such that u' and y satisfy (16), moreover vRu'. So, since u and y also satisfy (16), by the uniqueness, $u' = u$ and vRu.

Next, suppose $x \in \mathfrak{N}'$ and $v, u \in W$ such that $f(x) = v$ and vRu. We show that there exists $y \in \mathfrak{N}'$ such that $f(y) = u$ and $xR'y$.

The only interesting case to consider is $d(v) = 2$ and $u \neq v$. In this case $u = t$. Since $f(x) = v$, v and x satisfy (16), so

$$\mathfrak{N}', x \models \bigwedge prop(v) \wedge \Delta notprop(v) \wedge ((q \to r) \to q). \tag{17}$$

Note that

$$\vdash_{\mathbf{IPC}} ((q \to r) \to q) \to \neg\neg q.$$

Thus, $\mathfrak{N}', x \models \neg\neg q$, which means that there exists $y \in W'$ such that $xR'y$ and $\mathfrak{N}', y \models q$. Since

$$\mathfrak{N}', y \models \bigwedge prop(v) \wedge \Delta notprop(v),$$

we have that $\mathfrak{N}', y \models p_1 \wedge \cdots p_n$, i.e. $f(y) = u$.

The surjectivity of f follows in the same way as in the proof of theorem 3.3.

By applying Lemma 4.4, Lemma 4.6 and using the same argument as that in the proof of Theorem 3.3, we can show that for every $v \in X$, $f^{-1}(R(v)) = V'(\varphi'_v)$, which is an admissible set. Therefore by Lemma 3.1, we obtain $f^{-1}(X) \in \mathcal{P}'$.

Hence, f is a surjective (descriptive frame) p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Then since $\mathfrak{F} \cong \langle W, R, \mathcal{P} \rangle$, $\mathfrak{F}$ is a p-morphic image of $\langle W', R', \mathcal{P}' \rangle$, which is a generated subframe of $\mathfrak{G}$. ☐

Remark 4.8. *We have enough information to discuss the behavior of the φ'_{w_i} in the $n+1$-Henkin model. Assume $x \models r$ and $x \models \varphi'_w$ for some point w of $\mathcal{U}(n)$ with $d(w) \geq 2$ and some point x of the $n+1$-Henkin model. We will show that in that case $x \models \varphi'_t$. If that is not the case, then there exists a u such that $x \models \varphi'_u$ with least depth ≥ 2, i.e. $x \not\models \varphi'_{u_i}$ for all immediate successor u_i of u. This means that $f(x) = u$. By the surjectivity of the function f constructed in the proof of Lemma 4.7, we know that there exists a point $v \in R(w)$ such that $d(v) = 2$ and a $y \in R'(x)$ such that $f(y) = v$. From this it follows that $y \not\models \psi'_v$. On the other hand, from $y \models r$, it is easy to see that $y \not\models \psi'_v$, which is a contradiction. We have to conclude to $x \models \varphi'_t$.*

Of course, r can be false as well if φ'_t is true, and in the end it comes down to the following. The node φ'_w in the $n+1$-Henkin model generates a submodel consisting of an isomorphic copy of $\mathcal{U}(n)_w$ (here r is false) with above its top a copy of the Rieger-Nishimura ladder for r with $p_1, \cdots, p_n$ true everywhere. This also gives an indication how the p-morphism of the proof of Lemma 4.7 works in the case of the $n+1$-Henkin model for the submodel generated by φ'_w. On the bottom part it works as an isomorphism, the top part, i.e. the Rieger-Nishimura ladder is mapped onto a single point.

Now we are ready to prove Jankov's theorem on **KC**.

Theorem 4.9 (Jankov)**.** *If L is an intermediate logic such that $L \not\subseteq$ **KC**, then $L \vdash \theta$ and **IPC** $\nvdash \theta$ for some negation-free formula θ.*

Proof. We follow the idea of the proof of Theorem 3.4. Suppose χ is a formula satisfying

$$L \vdash \chi \text{ and } \mathbf{KC} \nvdash \chi.$$

Then there exists a finite rooted **KC**-frame $\mathfrak{F}$ with a largest element such that $\mathfrak{F} \not\models \chi$. By Lemma 3.2, there exists a model $\langle \mathfrak{F}, V \rangle$ on $\mathfrak{F}$ such that $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$ for some generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$. Note that $\mathcal{U}(n)_w$ has a largest element t, $t \models p_1 \wedge \cdots \wedge p_n$ and $col_V(v) \neq col_V(u)$ for all v, u in $\mathcal{U}(n)_w$.

Consider the formula ψ'_w. Suppose $L \nvdash \psi'_w$. Then there exists a descriptive frame $\mathfrak{G}$ of L such that $\mathfrak{G} \not\models \psi'_w$. By Lemma 4.7, $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. Thus, $\mathfrak{F}$ is an L-frame. Since $L \vdash \chi$, we have that $\mathfrak{F} \models \chi$, which leads to a contradiction.

Hence, $L \vdash \psi'_w$. We have that **IPC** $\nvdash \psi'_w$ by Corollary 4.3 and ψ'_w is negation-free, thus $\theta = \psi'_w$ is the required formula. $\square$

5 Some properties of $\mathcal{U}(n)$ and $\mathcal{H}(n)$

In this section we conclude in Theorem 5.1 the almost finished proof of section 2 that $\mathcal{U}(n)$ is isomorphic to the upper part of $\mathcal{H}(n)$. After that, we sharpen this result by giving a quick proof that these two models are even more "connected": every infinite upset of $\mathcal{H}(n)$ has an infinite intersection in $\mathcal{U}(n)$, or in other words, if an upset X generated by a point in the n-Henkin model has a finite intersection with its upper part, the n-universal model, then X lies completely in $\mathcal{U}(n)$. Both results were proved before in [1].

Theorem 5.1. *$Upper(\mathcal{H}(n))$ is isomorphic to $\mathcal{U}(n)$.*

Proof. Let $\mathcal{U}(n) = \langle U(n), R, V\rangle$. Define a function $f : \mathcal{U}(n) \to Upper(\mathcal{H}(n))$ by taking

$$f(w) = Cn(\varphi_w).$$

We show that f is an isomorphism. From the proof of Lemma 2.9 we know that

$$\mathcal{U}(n)_w \cong Upper(\mathcal{H}(n))_{f(w)}.$$

It then suffices to show that f is a bijection.

Let w, v be two distinct points of $\mathcal{U}(n)$. W.l.o.g. we may assume that $\neg wRv$, thus by Theorem 2.5, $\mathcal{U}(n), w \models \varphi_w$ but $\mathcal{U}(n), v \not\models \varphi_w$. We know from the proof of Lemma 2.9 that

$$\mathcal{U}(n)_w \cong Upper(\mathcal{H}(n))_{f(w)} \text{ and } \mathcal{U}(n)_v \cong Upper(\mathcal{H}(n))_{f(v)},$$

thus $Upper(\mathcal{H}(n))_{f(w)} \ncong Upper(\mathcal{H}(n))_{f(v)}$, so $f(w) \neq f(v)$.

For any point x in $Upper(\mathcal{H}(n))$, by Theorem 2.2, there exists a unique w_x such that $\mathcal{U}(n)_{w_x}$ is a p-morphic image of $Upper(\mathcal{H}(n))_x$, which by Corollary 2.8 implies that

$$Th(x) = Th(w_x) = Cn(\varphi_{w_x}),$$

therefore $f(w_x) = x$. $\square$

We call $w \in X$ a *border point* of an upset X of $\mathcal{U}(n)$, if $w \notin X$ and all successors v of w with $v \neq w$ are in X. Denote the set of all border points of X by $B(X)$. An upset X is uniquely characterized by its set of border points. Note that all endpoints $\mathcal{U}(n)$ which are not in X are in $B(X)$. The concept of border point was developed in studied in [2].

Fact 5.2. *If X is finite, then $B(X)$ is also finite.*

Proof. Since X is finite, there exists $k \in \omega$ such that $X \subseteq U(n)^k$. Observe that $B(X) \subseteq U(n)^{k+1}$, which means that $B(X)$ is finite, since $U(n)^{k+1}$ is finite. $\square$

The next lemma shows the syntactic side of the connection of upsets and their border points.

Lemma 5.3. *If $X = \{v_1, \cdots, v_k\}$ is a finite anti-chain in $\mathcal{U}(n)$ and $B(\mathcal{U}(n)_X) = \{w_1, \cdots, w_m\}$, then* $\vdash_{\mathbf{IPC}} (\varphi_{v_1} \vee \cdots \vee \varphi_{v_k}) \leftrightarrow (\psi_{w_1} \wedge \cdots \wedge \psi_{w_m})$.

Proof. In view of Theorem 2.3, it is sufficient to show that $\mathcal{U}(n) \models (\varphi_{v_1} \vee \cdots \vee \varphi_{v_k}) \leftrightarrow (\psi_{w_1} \wedge \cdots \wedge \psi_{w_m})$. By Theorem 2.5, it is then sufficient to show that

$$x \in R(v_1) \cup \cdots \cup R(v_k) \text{ iff } x \notin R^{-1}(w_1) \cup \cdots \cup R^{-1}(w_m).$$

For $\Rightarrow$: Suppose $x \in R(v_1) \cup \cdots \cup R(v_k) = U(n)_X$. If $x \in R^{-1}(w_i)$ for some $1 \leq i \leq m$, then since $U(n)_X$ is upward closed, we have that $w_i \in U(n)_X$, which contradicts the definition of $B(\mathcal{U}(n)_X)$.

For $\Leftarrow$: Suppose $x \notin R(v_1) \cup \cdots \cup R(v_k) = U(n)_X$. We show by induction on $d(x)$ that $x \in R^{-1}(w_i)$ for some $1 \leq i \leq m$.

$d(x) = 1$. Then x is an endpoint which is a border point. Thus, $x = w_i$ for some $1 \leq i \leq m$ and so $x \in R^{-1}(w_i)$.

$d(x) > 1$. The result holds trivially if x is a border point. Now suppose there exists $y \in R(x)$ such that $y \notin U(n)_X$. Since $d(y) < d(x)$, by the induction hypothesis, there exists $1 \leq i \leq m$ such that $y \in R^{-1}(w_i)$. Thus, $x \in R^{-1}(w_i)$. □

Theorem 5.4. *Let Γ be a point in $\mathcal{H}(n)$, i.e. Γ is an n-theory with the disjunction property. If $R(\Gamma) \cap \mathcal{U}(n)$ is finite, then $R(\Gamma) = R(\Gamma) \cap \mathcal{U}(n)$.*

Proof. Suppose $X = R(\Gamma) \cap \mathcal{U}(n)$ is finite. Then the set $B(X)$ of border points of X is finite. Let $B(X) = \{w_1, \cdots, w_m\}$. Suppose $\Gamma \nvdash \psi_{w_i}$ for some $1 \leq i \leq m$. Then there exists a descriptive frame $\mathfrak{G}$ such that $\mathfrak{G} \models \Gamma$ and $\mathfrak{G} \not\models \psi_{w_i}$. Since the underlying frame $\mathfrak{F}$ of $\mathcal{U}(n)_{w_i}$ is finite rooted, by Theorem 3.3, the latter implies that $\mathfrak{F}$ is a p-morphic image of a generated submodel of $\mathfrak{G}$. Thus, $\mathfrak{F} \models \Gamma$ and so $\mathcal{U}(n)_{w_i} \models \Gamma$, which is impossible since $w_i \in B(X)$ and $w_i \notin R(\Gamma) \cap \mathcal{U}(n)$.

Hence, we conclude that $\Gamma \vdash \psi_{w_i}$ for all $1 \leq i \leq m$. Let Y be the anti-chain consisting of all least points of X. Then by Lemma 5.3, $\Gamma \vdash \varphi_w$ for some $w \in Y$, which by Theorem 2.5 means that $\Gamma \in R(w)$, so $\Gamma \in \mathcal{U}(n)$, therefore $R(\Gamma) = R(\Gamma) \cap \mathcal{U}(n)$. □

References

[1] BEZHANISHVILI, N. *Lattices of Intermediate and Cylindric Modal Logics*. PhD thesis, University of Amsterdam, Netherlands, 2006.

[2] BEZHANISHVILI, N., AND DE JONGH, D. *Projective formulas in two variables in intuitionistic propositional logic, manuscript*, 2008.

[3] CHAGROV, A., AND ZAKHARYASCHEV, M. *Modal Logic*. Oxford University Press, 1997.

[4] DE JONGH, D. *Investigations on the Intuitionistic Propositional Calculus*. PhD thesis, University of Wisconsin, 1968.

[5] DE JONGH, D. A characterization of the intuitionistic propositional calculus. In *Intuitionism in Proof Theory*, M. Kino, Ed. North-Holland, 1970, pp. 211–217.

[6] JANKOV, V. The relationship between deducibility in the intuitionistic propositional calculus and finite implicational structures. *Soviet Mathematics Doklady* (1963).

[7] JANKOV, V. A. Ob ischislenii slabogo zakona iskluchennogo tret'jego. *Izvestija AN. SSSR, ser. matem. 32*, 5 (1968), 1044–1051.

[8] YANG, F. Intuitionistic subframe formulas, *NNIL*-formulas and n-universal models. Master's thesis, University of Amsterdam, 2009.