Information Theory Exercise Sheet #4 v2 (Process Convergence; Entropy Diagrams)

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1. **Branching Process** A random process repeatedly flips a fair coin to choose between the two words `AB` and `ABC`. A typical sample from this process is

   `A B C A B C A B A B A B C A B C A B C A B C A B C A B A B A B C A B A B A B C A B C ...`

   Compute the entropy rate of this process.

2. **Convolutional Process** Let $X_1, X_2, X_3, \ldots$ be fair coin flips, and define

   $$S_n \;=\; X_n + X_{n-1}.$$

   A typical sample from this process is

   $$1, 1, 1, 2, 1, 0, 1, 2, 2, 1, 0, 0, 0, 0, 0, 1, 1, 1, 2, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, \ldots$$

   Find the stationary distribution that describes the long-run behavior of $S_1, S_2, S_3, \ldots$.

3. **Entropy Diagrams** Let A, B, C be random variables such that

   $$\begin{aligned} I(A;B) &= 0, \\ I(A;C\,|\,B) &= I(A;B\,|\,C), \\ H(A\,|\,B,C) &= 0. \end{aligned}$$

   Find the relationship between $H(A)$ and $H(C)$.

4. **Guaranteed Corruption** A channel takes 8-bit words as input and outputs 8-bit words. For each input, the channel selects *exactly one* of the 8 bits at random and flips it.

   Construct an error-correcting code for this channel which *explicitly shows* that zero-error communication is possible over this channel at a rate of 5 bits per word.

5. **Increasing and Decreasing Uncertainty** For each of the three properties below, construct a (different) joint distribution over the variables X, Y, Z which satisfies the property.

(a) $H(X \mid Y = y) > H(X)$ for some y.

(b) $H(X \mid Y = y) < H(X)$ for all y.

(c) $I(X;Y) > I(X;Y \mid Z)$.

(d) $I(X;Y) < I(X;Y \mid Z)$.

6. **Implications** Let X, Y, Z be *binary* random variables such that $I(X;Y) = 0$ and $I(X;Z) = 0$.

   (a) Does it follow that $I(X;Y,Z) = 0$? If yes, prove it. If no, give a counterexample.

   (b) Does it follow that $I(Y;Z) = 0$? If yes, prove it. If no, give a counterexample.

7. **Bottleneck** Suppose that

$$A \rightarrow B \rightarrow C$$

   form a Markov chain (that is, A is independent of C given B). Suppose further that B only takes b different values. Prove that $I(A;C) \leq \log b$.

Homework is exercises $\{1, 4, 5\}$ or $\{1, 7, 5\}$.