Stopping Rules and Wald's Equality

Arianna Novaro

Information Theory

Master of Logic - UvA

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Overview

The Secretary Problem

The Secretary Problem

- Exactly one position available
- Known number n of applicants
- Sequentially and in random order
- Ranking is possible
- No second thougths
- Only the best is acceptable

Solution to the Secretary Problem

Strategy k

- Reject a certain number $k - 1$ of candidates.
- Select the first candidate who is better than all the previous ones.
- If all of them are worse, choose the last one ($=$ failure).

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Solution to the Secretary Problem

$B_j =$ the j-th candidate is the best
$S_j =$ the j-th candidate is selected

Probability of success with strategy k

$$P(B_j) = \frac{1}{n}$$

$$P(S_j|B_j) = \begin{cases} 0, & j < k \\ \frac{k-1}{j-1}, & j \geq k \end{cases}$$

$$P(B_j \cap S_j) = P(B_j) \times P(S_j|B_j) = \begin{cases} \frac{1}{n}, & k = 1 \\ \frac{k-1}{n} \sum_{j=k}^{n} \frac{1}{j-1}, & k \geq 2 \end{cases}$$

Stopping Times

Let $\mathbf{X} = \{X_n : n \geq 0\}$ be a stochastic process.

Random Time

A *random time* τ is a discrete random variable on the same probability space as $\mathbf{X}$, taking values in the time set $\mathbb{N} = \{0, 1, 2, \ldots\}$.

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Stopping Time

A *stopping time* with respect to $\mathbf{X}$ is a random time such that for each $n \geq 0$, the event $\{\tau = n\}$ is completely determined by (at most) the total information known up to time n, $\{X_0, \dots, X_n\}$.

Are these stopping rules?

- Playing until you reach the fifth gamble.
- Playing until you reach a total fortune of N\$ for the first time in the evening.
- Playing until you reach a total fortune of N\$ for the last time in the evening.
- Playing until you either reach a total fortune of N\$ or you run out of money.

Are these stopping rules?

- Playing until you reach the fifth gamble. ✓
- Playing until you reach a total fortune of N\$ for the first time in the evening. ✓
- Playing until you reach a total fortune of N\$ for the last time in the evening. ✗
- Playing until you either reach a total fortune of N\$ or you run out of money. ✓

Stopping Rules

- Playing until you reach the fifth gamble.

Independent case

τ is a random time that is independent of $\mathbf{X}$. In this case, $\{\tau = n\}$ doesn't depend *at all* on $\mathbf{X}$.

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First passage - Hitting Times

Suppose that $\mathbf{X}$ has a discrete state space and let i be a fixed state. The *first passage time* of the process into state i is:

$$\tau = \min\{n \geq 0 : X_n = i\}$$

Stopping Rules

$A = \{i\}$: collection of states

$$\tau = \min\{n \geq 0 : X_n \in A\}$$

$$\{\tau = n\} = \{X_0 \neq i, \ldots, X_{n-1} \neq i, X_n = i\}$$

Stopping Rules

$A = \{i\}$: collection of states

$$\tau = \min\{n \geq 0 : X_n \in A\}$$

$$\{\tau = n\} = \{X_0 \neq i, \ldots, X_{n-1} \neq i, X_n = i\}$$

Hitting times are stopping times

$$\{\tau = 0\} = \{X_0 \in A\}$$

$=$ it only depends on X_0.

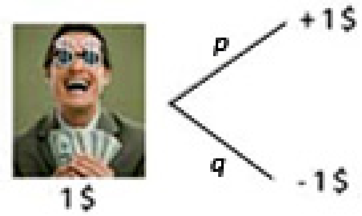
$$\{\tau = n\} = \{X_0 \notin A, \ldots, X_{n-1} \notin A, X_n \in A\}$$

$=$ it only depends on $\{X_0, \ldots, X_n\}$ for $n \geq 1$.

Gambler's Ruin Problem

- Playing until you either reach a total fortune of N\$ or you run out of money

First passage time to the set $A = \{0, N\}$.



Gambler's Ruin Problem

R_n: total fortune after the n^{th} gamble ($R_0 = i$).
$\{R_n : n \geq 0\}$:

$$R_n = \Delta_1 + \cdots + \Delta_n$$

$$\tau_i = \min\{n \geq 0 : R_n \in \{0, N\} | R_0 = i\}$$

- If $R_{\tau_i} = N$ you WIN.
- If $R_{\tau_i} = 0$ you GET RUINED.

$$P_i = P(R_{\tau_i} = N) = pP_{i+1} + qP_{i-1}$$

$$P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N}, & \text{if} p \neq q \\ \frac{i}{N}, & \text{if} p = q = \frac{1}{2} \end{cases}$$

Abraham Wald (1902 - 1950)

"He was a master at deriving complicated results in amazingly simple ways." (Johnson & Kotz, 1997)

Wald's Equality

Let $\{X_n : n \geq 1\}$ be a sequence of IID RVs, each of expectation $\mathbb{E}[X]$. If τ is a stopping time for $\{X_n : n \geq 1\}$ and if $\mathbb{E}[\tau] < \infty$, then the sum $S_\tau = X_1 + X_2 + \cdots + X_\tau$ at the stopping time τ satisfies:

$$\mathbb{E}[S_\tau] = \mathbb{E}[X]\mathbb{E}[\tau]$$

Wald's Equality

$$
\begin{aligned}
\mathbb{E}[S_\tau] =& \mathbb{E}\left[\sum_n X_n \mathbb{I}_{\tau \geq n}\right] = \\
&= \sum_n \mathbb{E}\left[X_n \mathbb{I}_{\tau \geq n}\right] = \\
&= \sum_n \mathbb{E}[X_n] \cdot \mathbb{E}[\mathbb{I}_{\tau \geq n}] = \\
&= \mathbb{E}[X] \sum_n \mathbb{E}\left[\mathbb{I}_{\tau \geq n}\right] = \\
&= \mathbb{E}[X] \sum_n Pr\{\tau \geq n\} = \\
&= \mathbb{E}[X]\mathbb{E}[\tau]
\end{aligned}
$$

Further readings and References

- Blackwell D. (1946), "On an Equation of Wald", *Ann. Math. Statist.*, 17, 1: 84-87.
- Ferguson T. (1989), "Who solved the Secretary Problem?", *Statistical Science*, 4, 3: 282 - 289.
- Sigman K. (2009), *Stopping Times*, 1 - 6.
(available at: http://www.columbia.edu/ ks20/stochastic-I/stochastic-I-ST.pdf)

Thank you!