Information Theory Exercise Sheet #7

University of Amsterdam, Master of Logic, Fall 2014
Lecturer: Christian Schaffner
TA: Philip Schulz

Out: Wednesday, 10 December 2014 2014
(due: Friday, 26 December 2014, 17:00)

Previous Exam Questions

For each value of $m = 0, 1, 2, \ldots$, what is the capacity of the channel consisting of a BSC(ε) together with m symbols which are all transmitted perfectly? The transition matrix of this channel is given by:

$$\left[\begin{array}{cc|c} 1-\varepsilon & \varepsilon & 0\ldots0 \\ \varepsilon & 1-\varepsilon & 0\ldots0 \\ \hline 0 & 0 & \\ \vdots & \vdots & 1_{m\times m} \\ 0 & 0 & \end{array}\right]$$

where $1_{m\times m}$ is the m by m identity matrix with 1's on the diagonal and 0's everywhere else.

To be solved in Class

1. *Infinite entropy.* Find a discrete probability distribution $P_X(n)$ with infinite entropy $H(X) = \infty$.
   Hint: Recall the integral test for convergence from calculus, saying that the infinite sum $\sum_{n=N}^{\infty} f(n)$ converges to a real number if and only if the integral $\int_N^\infty f(x)dx$ is finite. Use the integral test to show that the infinite sum $\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$ diverges, whereas $\sum_{n=2}^{\infty} \frac{1}{n\log^2(n)}$ converges.
   Another Hint: Differentiate the functions $f(x) = \log(\log(x))$ and $g(x) = \frac{1}{\log(x)}$.

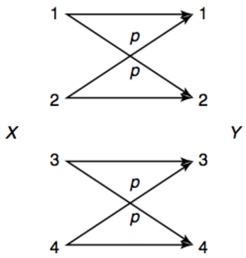
2. Compute $\frac{d}{dp}h(p)$.

Homework

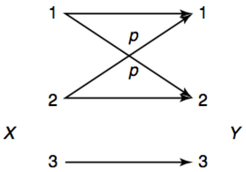
1. *Choice of channels.* Find the capacity C of the union of two (possibly different) channels $(\mathcal{X}_1, P_{Y_1|X_1}, \mathcal{Y}_1)$ and $(\mathcal{X}_2, P_{Y_2|X_2}, \mathcal{Y}_2)$, where at each time, one can send a symbol over channel 1 or channel 2 but not both. Assume that the output alphabets are distinct and do not intersect.

   (a) [6 points] Show that $2^C = 2^{C_1} + 2^{C_2}$.
   Use this result to calculate the capacity of the following channels:

(b) [2 points] Two parallel BSCs:



(c) [2 points] BSC and a single symbol:



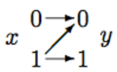
2. *Capacities.* Find the capacity and optimal input distribution of the following channels:

   (a) [3 points] The ternary channel with transition matrix

$$P_{Y|X} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

   (b) [6 points] The Z-channel with transition matrix

$$P_{Y|X} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$



3. Let the input to a channel be a word of 8 bits. The output is also a word of 8 bits. Each time it is used, the channel flips exactly one of the transmitted bits, but the receiver does not know which one. The other seven bits are received without error. All 8 bits are equally likely to be the one that is flipped.

   (a) [2 points] Determine the capacity of this channel.

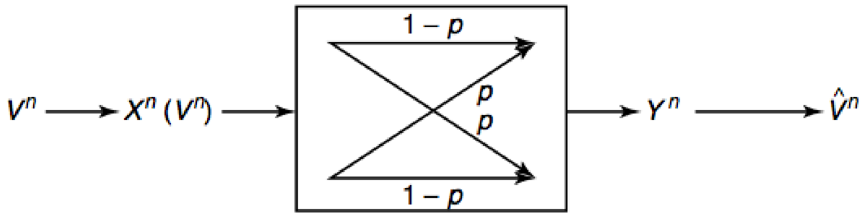
   (b) [4 points] Show, by describing an *explicit encoder and decoder* that the it is possible to communicate with *zero error* 5 bits per channel use.
   Hint: Extend the (7,4) Hamming code to a (8,5) code that does the job.

4. *Noise alphabets.* Consider an additive-noise channel where $\mathcal{X} = \{0, 1, 2, 3\}$ and $Y = X + Z$ with addition over the integers $\mathbb{Z}$. Z is uniformly distributed over three distinct integer values $\mathcal{Z} = \{z_1, z_2, z_3\}$.

   (a) [2 points] What is the maximum capacity over all choices of the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

   (b) [4 points] What is the minimum capacity over all choices of the $\mathcal{Z}$ alphabet? Give distinct integer values z_1, z_2, z_3 and a distribution on $\mathcal{X}$ achieving this.

5. [2 points] *Source and channel.* We wish to encode a sequence $V_1, V_2, \ldots$ of iid variables with distribution $P_V(0) = \alpha$, $P_V(1) = 1 - \alpha$ for transmission over a binary symmetric channel with crossover probability p.



Find conditions on α and p so that the probability of error $\Pr[\hat{V}^n \neq V^n]$ can be made to go to zero as $n \to \infty$.