Information & Communication Exercise Sheet #1

University of Amsterdam, Bachelor of Computer Science, January 2015
Lecturer: Christian Schaffner

Out: Monday, 5 January 2015
(due: Monday, 12 January 2015, 9:00)

To be solved in Class

1. **Probability theory** Prove Bayes' theorem.

   Theorem 1 (Bayes' theorem) *Let E_1 and E_2 be probability events with $P[E_2] \neq 0$. Then,*

   $$P[E_1|E_2] = \frac{P[E_1] \cdot P[E_2|E_1]}{P[E_2]} .$$

2. Prove the *union bound* which states that for arbitrary events E_1, E_2, we have

   $$P[E_1 \cup E_2] \leq P[E_1] + P[E_2] .$$

3. ([MacKay, Ex 2.31]) A circular coin of diameter a is thrown onto a square grid whose squares are $b \times b$. ($a < b$) What is the probability that the coin will lie entirely within one square?

4. What is the probability that two (or more) students in our class have the same birthday? Let us assume that everybody was born in the same year.

5. ([MacKay] Example 2.3:) Jo has a test for a nasty disease. We denote Jo's state of health by the random variable A ($A = 1$ is Jo has the disease and $A = 0$ if not) and the test result by B ($B = 1$ if the test is positive and $B = 0$ if the test is negative).

   The test is 95% reliable: in 95% of cases of people who really have the disease, a positive result is returned, and in 95% of cases of people who do not have the disease, a negative result is obtained. Finally, 1% of people of Jo's age and background have the disease.

   If Jo has the test and it is positive, what is the probability that Jo has the disease?

6. **Expected values** Let X and Y be two real random variables with joint distribution P_{XY}.

   (a) Show that expected values are linear: For arbitrary real numbers $a, b \in \mathbb{R}$, it holds that

   $$\mathbb{E}_{XY}[aX + bY] = a\, \mathbb{E}_X[X] + b\, \mathbb{E}_Y[Y] \ .$$

   (b) Show that if X and Y are independent, it holds that

   $$\mathbb{E}_{XY}[X \cdot Y] = \mathbb{E}_X[X] \cdot \mathbb{E}_Y[Y] \ .$$

   Give an example of a joint distribution P_{XY} for which $\mathbb{E}_{XY}[X \cdot Y] \neq \mathbb{E}_X[X] \cdot \mathbb{E}_Y[Y]$.

7. ([MacKay] Exercises 2.21 onwards) Maybe you are not so comfortable with computing expectations in cases where the function $f(x)$ depends on the probability $P_X(x)$. The next few examples address this concern.

(a) Let $P_X(a) = 0.1$, $P_X(b) = 0.2$, and $P_X(c) = 0.7$. Let $f(a) = 10$, $f(b) = 5$, and $f(c) = 10/7$. What is $\mathbb{E}_X[f(X)]$? What is $\mathbb{E}_X[1/P_X(X)]$?

(b) What is the probability that $P_X(X) \in [0.15, 0.5]$? What is

$$\Pr\left[\left|\log \frac{P_X(X)}{0.2}\right| > 0.05\right] ?$$

(c) For an arbitrary distribution Q_Y over $\mathcal{Y}$, what is $\mathbb{E}_Y[1/Q_Y(Y)]$?

Homework

1. **Email** Please send an email to Chris (c.schaffner@uva.nl) stating your name, the program and year you are following (e.g. 3rd year Bachelor of Informatica), and (at least) one sentence about your motivation to follow this course. `1 p.`

2. (a) Compute the number of possible orders of a deck of 52 playing cards (assuming that you intend to draw one card after the other without replacement). `1 p.`

   (b) Now suppose we have a perfectly shuffled big deck, consisting of two *identical* decks of 52 cards (so 104 cards in total). In how many different ways can the big deck be ordered? `2 p.`

3. Prove the following inequality for real numbers $p_1, p_2, \ldots, p_n \in [0, 1]$: `3 p.`

$$(1 - p_1)(1 - p_2) \cdots (1 - p_n) \geq 1 - p_1 - p_2 - \ldots - p_n \,.$$

   Hint: For an event E, the event $\bar{E}$ is the event that E does not occur, hence $\Pr[\bar{E}] = 1 - \Pr[E]$. Consider *independent* events E_i with probabilities $p_i = P[E_i]$ and use the union bound.

4. The mean of a random variable X is $\mu = \mathbb{E}[X]$. The variance of X is defined as $\mathrm{Var}[X] = \mathbb{E}[(X - \mu)^2]$. The *standard deviation* of X is $\sigma[X] = \sqrt{\mathrm{Var}[X]}$.

   (a) Show that $\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. `1 p.`

   (b) Show that for any real $a > 0$, it holds that $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$, and $\mathrm{Var}[X + a] = \mathrm{Var}[X]$. `2 p.`

   (c) Show that for independent random variables X, Y, we have $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$. `1 p.`

   (d) Let X be a random variable with *Bernoulli distribution* $P_X(1) = p$ and $P_X(0) = 1 - p$. Compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$. `2 p.`

   (e) Let Y be a random variable with *binomial distribution* $P_Y(y) = \binom{n}{y} p^y (1 - p)^{n-y}$. Compute $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$. `2 p.`

5. An urn contains K balls, of which B are black and $W = K - B$ are white. Fred draws a ball at random from the urn and replaces it, N times.

   (a) What is the probability distribution of the number of times a black ball is drawn, n_B? `1 p.`

   (b) What is the expectation of n_B? What is the variance of n_B? What is the standard deviation of n_B? Give numerical answers for the cases $N = 5$ and $N = 400$, when $B = 2$ and $K = 10$. `4 p.`