Yao's Garbled Circuit

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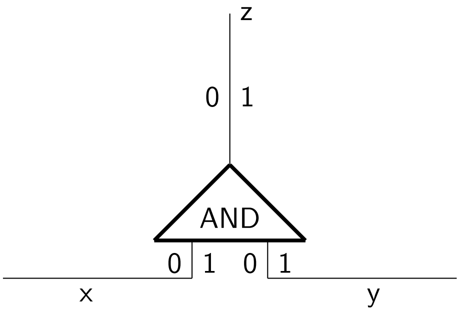
Based on a Boolean circuit.

How does it work?

Transform the function $f(x, y)$ into a garbled boolean circuit $C(x, y)$.

- A garbled boolean circuit is a collection of garbled boolean gates.
- First understand what a single garbled gate is and then easily generalize to the entire garbled circuit.

Garbled Gate: What is that?



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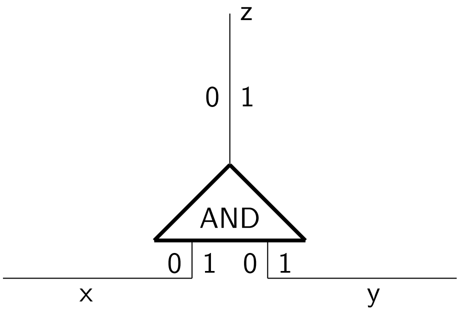

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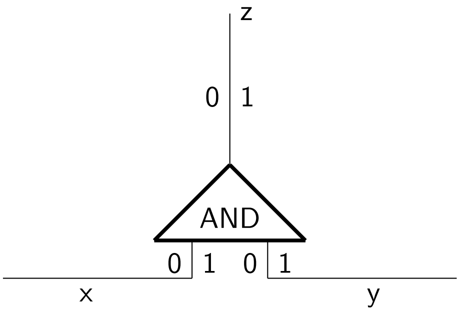

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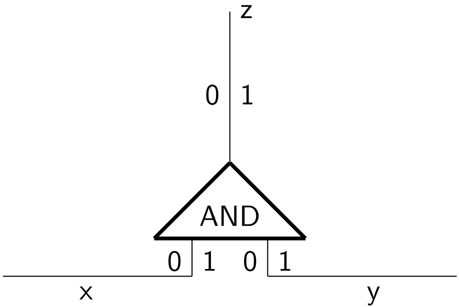

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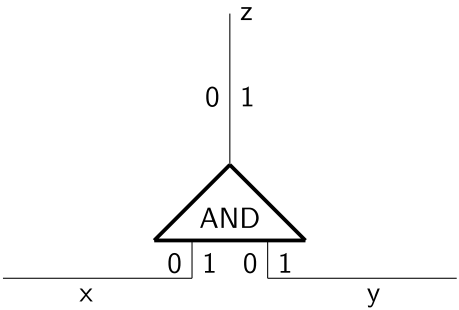

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- For each of these three wires we have two values: $0, 1$ (input values for the input wires, output values for the output wire).
- Once input values a, b are selected, we want to compute $g(a, b)$ securely.

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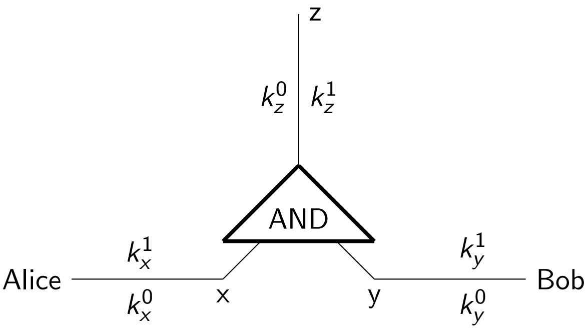

- We need to "associate" k_z^0, k_z^1 with $k_x^0, k_x^1, k_y^0, k_y^1$. We will do that using the Garbled Computation Table (GCT).

Garbled Computation Table (GCT)

- View $k_z^0, k_z^1, k_x^0, k_x^1, k_y^0, k_y^1$ as encryption keys and encrypt k_z^0, k_z^1 under appropriate pairs of input keys.

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k_x^0	k_y^1	k_z^0	$E_{k_x^0}(E_{k_y^1}(k_z^0))$
k_x^1	k_y^0	k_z^0	$E_{k_x^1}(E_{k_y^0}(k_z^0))$
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k_x^1	k_y^1	k_z^1	$E_{k_x^1}(E_{k_y^1}(k_z^1))$

- NOTE: Given two input keys k_x^a and k_y^b, only one row of the GCT can be decrypted correctly, namely: $E_{k_x^a}(E_{k_y^b}(k_z^{g(a,b)}))$.

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- Bob still needs his own key to decrypt the GCT so Alice must send it to him.
- If Alice sends both k_y^0, k_y^1 to Bob, then Bob can decrypt more and if Bob asks for Alice the key that corresponds to his input, then Alice learns Bob's input.

Oblivious Transfer

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By using Oblivious Transfer. A protocol in which:

- sender inputs x_0, x_1
- receiver inputs s
- receiver obtains x_s

- Sender learns nothing about s, receiver learns nothing about x_{s-1}.

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Sends output $k_z^{g(b',b)}$ to Alice and the computation of the garbled gate is over.

Generalizing to a circuit

$E_{k^0_{w_1}}(E_{k^0_{w_2}}(k^0_{w_3}))$
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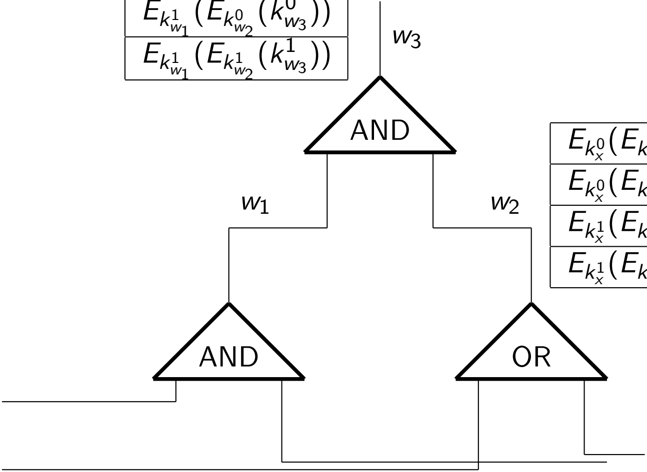

$E_{k^0_x}(E_{k^0_y}(k^0_z))$
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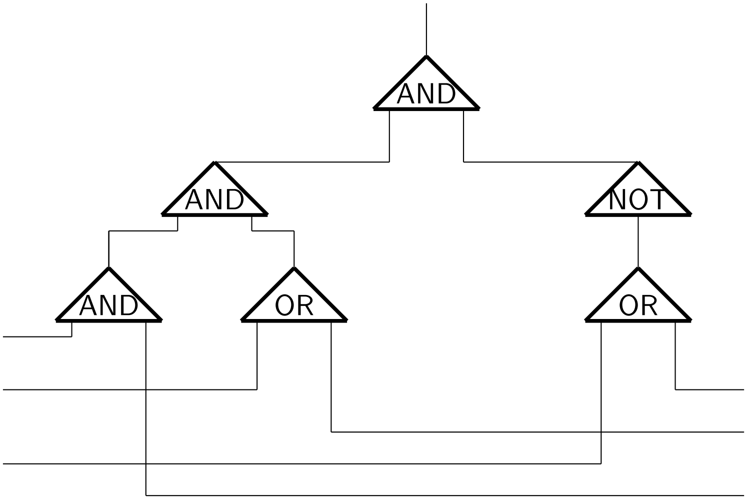

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P_2 computes the circuit and sends the output back to P_1.

Simplifying the above

- Two padlock keys for each wire.
- 4 doubly-locked boxes for each gate, such that:
  - locked in a way that a pair of keys can open only one
  - in each box there is a new key
- Open boxes one at a time until final output.
- No other information revealed (keys can not be associated with 0 or 1).
- Actual Construction: Replace each box with a row in GCT and physical keys with encryption keys

Proof of Security

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- Yao's protocol is secure in the presence of semi-honest adversaries: follow the protocol, but attempt to learn information by analyzing the transcript of messages during the execution.
- Took 20 years to provide a full formal proof.
- Formal proof definitely not trivial.
- Proof can be found in the paper posted in the website of the course under Yao's garbled circuits and there is a proper citation of it in my handout.

Interesting Question

How does Bob know when he has decrypted correctly?
We use serial encryption, meaning that if Bob decrypts the GCT, he will obtain 4 random values. How can he distinguish which one of these is the correct one?
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- In this way, only the correct keys give a redundant block.
- For instance while encrypting the key, also encrypt 100 zeros.
- Then, the correct value is followed by 100 zeros.
- Probability that an incorrect value is followed by 100 zeros is negligible.