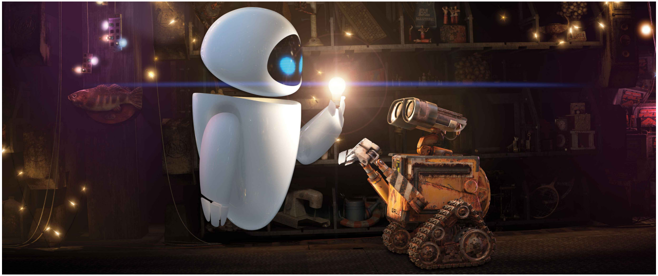
Zero Knowledge Proofs

MoL Research Project



Maaike Zwart & Suzanne van Wijk

January 2015

Institute for Logic, Language and Computation

Overview

In this project:

Lectures:

- Complexity Theory
  - Turing Machines
  - Classes P and NP
  - Interactive proofs
- Zero Knowledge Proofs
  - ZK proofs, intuitive
  - ZK proofs
  - ZK proofs for NP problems
  - Commitment scemes

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In this project:

Work for you:

- 3 exercise classes
  - Classes right after the lectures (with a 1 hour lunch break in between)
  - Hand in exercises the next day before 17:00 h
- 1 final assignment
  - Presentations on 22 January
  - You may work in pairs
- Grading: pass/fail

Historical context

- $\sim$ 1900 - 1930: formalising mathematics
  Hilbert: *Wir müssen wissen, wir werden wissen*
- 1931: Gödel's incompleteness theorems
- 1935 - 1936: Entscheidungsproblem solved by Church and Turing
  - Methods similar to Gödel
  - Needed to formalise 'computation' (Church - Turing Thesis)
  - Turing's solution: Turing machines.
- From this concept, modern computers were developed.

Alan Turing 1912 - 1954

- Founder of Computer Science and Artificial Intelligence
- Turing Test
- Cracked the Enigma Code
- Died a tragic death

A Theoretical Model for a Computer

stephengg, Flickr

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Therefore they are used to:

- Find limits of computability
- Analyse algorithms: how long does a computation take?

Turing Machines

Turing Machines

	0	0	0	0	0	0	1	0	0	1	1	1	0	0	0	1	0	1

Turing Machines

	0	0	0	0	0	0	1	0	0	1	1	1	0	0	0	1	0	1

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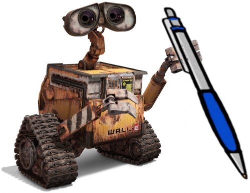

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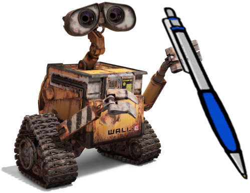

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	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	1	0	1
	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

Turing Machines

	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0	1	0	1
	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Turing Machines

inputtape	0	0	0	0	0	0	1	0	0	1	0	1	0	0	0
worktape	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
worktape	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
outputtape	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Input: x
Output: M(x)

Programming a Turing Machine

A Turing Machine needs instructions: write what? move where? These instructions depend on what the TM reads, and on its internal *state*. Notice: also need instruction: next state?

Suppose I want a TM changes the first two 1's it encounters to a 0, and then stops.

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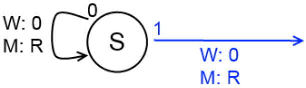

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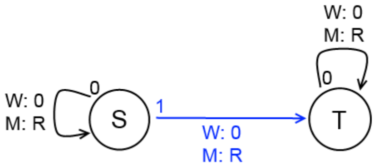

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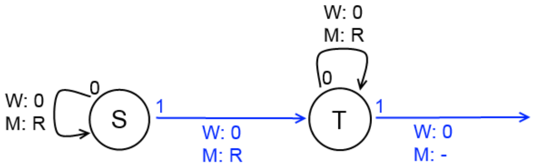

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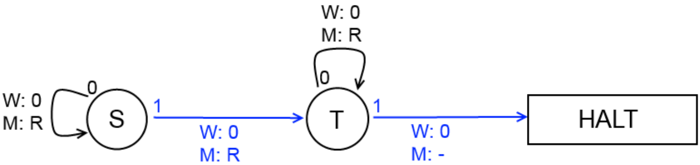

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- $\delta : Q \times \Gamma \to \Gamma \times \{L, R, -\} \times Q$, the transition function, defining which symbol to write, where to move and which is the next state, depending on the current state and the symbol being read.

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Turing Machines example: counting

Let $\Gamma = \{0, 1, blank\}$, $Q = \{S, 1, 2, halt\}$, and δ defined by:

State?	read?	write:	move:	next state:
state S	blank	blank	L	1
	0	0	R	0
	1	1	R	0
state 1	blank	1	R	2
	0	1	L	2
	1	0	L	1
state 2	blank	blank	R	halt
	0	0	L	2
	1	1	L	2

Movie of a working Turing Machine

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Catalog problems according to 'how hard they are to solve'.

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Find algorithm: build a TM M such that:

- $x \in L \iff M(x) = 1$
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Note: Complexity of a problem is independent of the model of TM (Exercise!)

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P: example

Is x a palindrome?

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Time to compute solution: $\sim 2^{|\phi|}$

Exponential!

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Exponential!

Witness: an assignment to the variables.
Time to check assignment: $\sim |\phi|$

P vs NP

Clearly $P \subseteq NP$
Open problem: $P = NP$?

General consensus: no.
We will use this later on!

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Fun Facts:

- Solving the P vs NP problem wins you a million dollars!
- If P = NP ... could be a problem:
  - Digital security collapses
  - Mathematicians would be out of work

But, as Donald Knuth[1] reassures us: A proof of P = NP will almost certainly be non-constructive, so no worries there :)

[1] Knuth, Donald E. (May 20, 2014). "Twenty Questions for Donald Knuth". informit.com.