# WISC is independent from ZF

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**WISC** (which stands for **W**eakly **I**nitial **S**et of **C**overs) is the following statement:

> For every set $X$ there is a set of surjections $\{p_i\colon Y_i \to X\}$ onto $X$ such that for every surjection $q\colon Z \to X$ there is an $i \in I$ and a function $f\colon Y_i \to Z$ such that $p_i = q \circ f$.

**WISC** is implied by the axiom of choice and the existence of enough projectives. For more information on **WISC**, see [1] (where it is called **AMC**) and the nLab.

The aim of this note is to prove that **WISC** is independent from **ZF**. The argument is inspired by the work of Andreas Blass [2].

**Theorem 0.1** **ZF** *is consistent with the statement that every limit ordinal has cofinality* $\omega$.

**Proof.** By a celebrated result of Gitik [3] **ZF** is consistent with the statement that all uncountable cardinals are singular, or, in other words, that $\omega$ is the only regular cardinal. Because the cofinality of a limit ordinal is a regular cardinal, **ZF** is consistent with the statement that $\omega$ is the cofinality of every limit ordinal. □

**Theorem 0.2** **WISC** *is not provable in* **ZF**.

**Proof.** Suppose there is a family $(p_i\colon B_i \to \mathbb{N} \mid i \in I)$ of surjections onto $\mathbb{N}$ such that every surjection onto $\mathbb{N}$ is refined by one in this family. We will show that this is incompatible with the statement that every limit ordinal has cofinality $\omega$.

Put $A = I + \{0, 1\}$, $B_0 = \emptyset$, $B_1 = \{0\}$ and let

$$f\colon \sum_{a \in A} B_a \to A$$

be the obvious projection. We will write $W = W(f)$ for the W-type associated to $f$.

By transfinite induction on $W$, we can now define a map $m\colon W \to Ord$ as follows:

$$\begin{aligned} m(\sup_i(t)) &= \sup\{m(tb) : b \in B_i\}, \\ m(\sup_0(t)) &= 0, \\ m(\sup_1(t)) &= t(0) + 1. \end{aligned}$$

Observe that:

1. 0 lies in the image on $m$.
2. The image of $m$ is closed under taking successor ordinals.
3. The image of $m$ is closed under countable suprema: for if $\beta = \sup(\alpha_n)$ and $\alpha_n \in m(W)$ for every $n \in \mathbb{N}$, then

   $$(\forall n \in \mathbb{N})\,(\exists w \in W)\, m(w) = \alpha_n.$$

   Weak initiality of $(f_i\colon B_i \to \mathbb{N} \mid i \in I)$ now implies that there is an $i \in I$ and a function $t\colon B_i \to W$ such that

   $$(\forall b \in B_i)\, m(tb) = \alpha_{p_i(b)}.$$

   Hence $\beta = m(\sup_i(t))$.

So if every limit ordinal had cofinality $\omega$, the map $m$ would be surjective. Contradiction. □

**Remark 0.3** By refining this argument one can show that **WISC** implies (in **ZF**) that there are class many regular cardinals and hence (by Proposition 2 on page 151 of [2]) that every variety has free algebras.

# References


[1] B. van den Berg and I. Moerdijk. The axiom of multiple choice and models for constructive set theory. arXiv:1204.4045, 2012.

[2] A.R. Blass. Words, free algebras, and coequalizers. *Fund. Math.*, 117(2):117–160, 1983.

[3] M. Gitik. All uncountable cardinals can be singular. *Israel J. Math.*, 35(1-2):61–88, 1980.