5th Homework sheet Proof Theory

- Deadline: 16 March, 13:00 sharp.
- Submit your solutions by handing them to the lecturer or the teaching assistant at the *beginning of the exercise class*.
- There are some exercises on the other page.
- Good luck!

A formula φ is in *negation normal form* if any implication occurring in φ has an atomic formula on the left and $\bot$ on the right. In classical logic every formula is equivalent to a formula in negation normal form. One way of seeing this is as follows: define for any formula φ two new formulas $\mathbf{T}\varphi$ and $\mathbf{F}\varphi$ by simultaneous recursion, as follows:

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\begin{array}{lcll@{\qquad}lcll}
\mathbf{T}\varphi & = & \varphi & \text{if } \varphi \text{ is atomic} & \mathbf{F}\varphi & = & \neg\varphi & \text{if } \varphi \text{ is atomic} \\
\mathbf{T}(\varphi \wedge \psi) & = & \mathbf{T}\varphi \wedge \mathbf{T}\psi & & \mathbf{F}(\varphi \wedge \psi) & = & \mathbf{F}\varphi \vee \mathbf{F}\psi & \\
\mathbf{T}(\varphi \vee \psi) & = & \mathbf{T}\varphi \vee \mathbf{T}\psi & & \mathbf{F}(\varphi \vee \psi) & = & \mathbf{F}\varphi \wedge \mathbf{F}\psi & \\
\mathbf{T}(\varphi \to \psi) & = & \mathbf{F}\varphi \vee \mathbf{T}\psi & & \mathbf{F}(\varphi \to \psi) & = & \mathbf{T}\varphi \wedge \mathbf{F}\psi & \\
\mathbf{T}(\exists x\, \varphi) & = & \exists x\, \mathbf{T}\varphi & & \mathbf{F}(\exists x\, \varphi) & = & \forall x\, \mathbf{F}\varphi & \\
\mathbf{T}(\forall x\, \varphi) & = & \forall x\, \mathbf{T}\varphi & & \mathbf{F}(\forall x\, \varphi) & = & \exists x\, \mathbf{F}\varphi &
\end{array}
$$

It is easy to see that for any formula φ both $\mathbf{T}\varphi$ and $\mathbf{F}\varphi$ are in negation normal form and that $\mathbf{T}\varphi$ is classically equivalent to φ, while $\mathbf{F}\varphi$ is classically equivalent to $\neg\varphi$ (you do not need to prove these facts).

(a) *(40 points)* Show that for every formula φ the sequent $\mathbf{T}\varphi, \mathbf{F}\varphi \Rightarrow \bot$ is derivable in the intuitionistic sequent calculus without the cut rule.

(b) *(40 points)* Prove the following implication: if the sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$ is provable in the classical sequent calculus for predicate logic without the cut rule, then the sequent $\mathbf{T}\varphi_1, \ldots, \mathbf{T}\varphi_n, \mathbf{F}\psi_1, \ldots, \mathbf{F}\psi_m \Rightarrow \bot$ is derivable in intuitionistic sequent calculus for predicate logic without the cut rule.

(c) *(20 points)* Define $\varphi^* = \neg\mathbf{F}\varphi$. Deduce from (b) that φ is a classical tautology precisely when φ^* is an intuitionistic tautology.

(d) *(Chocolate bar exercise)* Is the mapping $\varphi \mapsto \varphi^*$ defined in (c) a negative translation? Justify your answer!