CHAPTER 13

Sequent calculus for predicate logic

1. Classical sequent calculus

The axioms and rules of the classical sequent calculus are:

$$\text{Axioms} \quad \begin{cases} \Gamma, \varphi \Rightarrow \Delta, \varphi & \text{for atomic } \varphi \\ \Gamma, \bot \Rightarrow \Delta \end{cases}$$

	Left	Right
$\wedge$	$\frac{\Gamma,\alpha_1,\alpha_2\Rightarrow\Delta}{\Gamma,\alpha_1\wedge\alpha_2\Rightarrow\Delta}$	$\frac{\Gamma\Rightarrow\beta_1,\Delta \qquad \Gamma\Rightarrow\beta_2,\Delta}{\Gamma\Rightarrow\beta_1\wedge\beta_2,\Delta}$
$\vee$	$\frac{\Gamma,\beta_1\Rightarrow\Delta \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\vee\beta_2\Rightarrow\Delta}$	$\frac{\Gamma\Rightarrow\alpha_1,\alpha_2,\Delta}{\Gamma\Rightarrow\alpha_1\vee\alpha_2,\Delta}$
$\rightarrow$	$\frac{\Gamma\Rightarrow\Delta,\beta_1 \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\rightarrow\beta_2\Rightarrow\Delta}$	$\frac{\Gamma,\alpha_1\Rightarrow\alpha_2,\Delta}{\Gamma\Rightarrow\alpha_1\rightarrow\alpha_2,\Delta}$
$\forall$	$\frac{\Gamma,\varphi[t/x]\Rightarrow\Delta}{\Gamma,\forall x\,\varphi\Rightarrow\Delta}$	$\frac{\Gamma\Rightarrow\Delta,\varphi}{\Gamma\Rightarrow\Delta,\forall x\,\varphi}$
$\exists$	$\frac{\Gamma,\varphi\Rightarrow\Delta}{\Gamma,\exists x\,\varphi\Rightarrow\Delta}$	$\frac{\Gamma\Rightarrow\Delta,\varphi[t/x]}{\Gamma\Rightarrow\Delta,\exists x\,\varphi}$

The side condition in $\exists L$ and $\forall R$ is that the variable x may not occur freely in Γ or Δ (this variable is sometimes called an *eigenvariable*).

In addition, we have the following cut rule:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

We outline a proof of cut elimination. First a definition:

Definition 1.1. The logical depth $\mathrm{dp}(\varphi)$ of a formula is defined inductively as follows: the logical depth of an atomic formula is 0, while the logical depth of $\varphi\Box\psi$ is $\max(\mathrm{dp}(\varphi), \mathrm{dp}(\psi))+1$. Finally, the logical depth of $\forall x\,\varphi$ or $\exists x\,\varphi$ is $\mathrm{dp}(\varphi)+1$. The rank $\mathrm{rk}(\varphi)$ of a formula φ will be defined as $\mathrm{dp}(\varphi)+1$.

If there is an application of the cut rule

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta},$$

then we call φ a *cut formula*. If π is a derivation, then we define its *cut rank* to be 0, if it contains no cut formulas (i.e., is *cut free*). If, on the other hand, it contains applications of the cut rule, then we define the cut rank of π to be the rank of any cut formula in π which has greatest possible rank.

LEMMA 1.2. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a derivation π and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is derivable as well. In fact, the latter has a derivation π' with a cut rank and size no greater than that of π.*

LEMMA 1.3. (Inversion Lemma) *Apart from the rules introducing $\forall$ on the left and $\exists$ on the right, each of the rules in the classical sequent calculus is* invertible*: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by a rule different from $\forall L$ and $\exists R$, then there are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π.*

In addition we have:

LEMMA 1.4. (Substitution Lemma) *If π is a derivation of $\Gamma \Rightarrow \Delta$, then there is a derivation of $\Gamma[t/x] \Rightarrow \Delta[t/x]$ which has no greater cut rank or size than π.*

PROOF. First of all, by a suitable renaming of the variables we may assume that x is never an eigenvariable; then systematically replace every occurrence of x in π by t. □

As before, the key step in the proof for cut elimination is the following:

LEMMA 1.5. (Key Lemma) *Suppose π is a derivation which ends with an application of the cut rule applied to a formula of rank d, while the rank of any other cut formula in π is strictly smaller than d. Then π can be transformed into a derivation π' with the same endsequent as π and which has cut rank strictly less than d.*

PROOF. The idea is to look at the structure of the cut formula. We know what to do when it is an atomic formula or its main connective is propositional, because then we proceed as we did in the chapter on propositional logic. In case the cut formula is of the form $\exists x\, \varphi$ or $\forall x\, \varphi$, then we proceed as we did in the intuitionistic case for implication. Since the cases are perfectly dual, we may assume that the last step was:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma \Rightarrow \forall x\, \varphi, \Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \Gamma, \forall x\, \varphi \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \Delta}$$

We now do an induction on the depth of the derivation π to show that we can get a derivation of $\Gamma \Rightarrow \Delta$ with the cut rank below d. We make a case distinction on what was the last rule which was applied in $\mathcal{D}_1$; the most tricky case is where the final rule in $\mathcal{D}_1$ introduced $\forall x\, \varphi$, while at the same time it was already present, like this:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma \Rightarrow \forall x\, \varphi, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}_1' \\ \Gamma, \forall x\, \varphi, \varphi(t) \Rightarrow \Delta\end{matrix}}{\Gamma, \forall x\, \varphi \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

By weakening there is also a derivation $\mathcal{D}_0'$ of $\Gamma, \varphi(t) \Rightarrow \forall x\, \varphi, \Delta$ which has no greater cut rank or size than $\mathcal{D}_0$, so we can apply the induction hypothesis on the proof

$$\frac{\begin{array}{cc} \mathcal{D}_0' & \mathcal{D}_1' \\ \Gamma, \varphi(t) \;\Rightarrow\; \forall x\, \varphi, \Delta & \Gamma, \forall x\, \varphi, \varphi(t) \;\Rightarrow\; \Delta \end{array}}{\Gamma, \varphi(t) \;\Rightarrow\; \Delta}$$

to obtain a derivation $\mathcal{D}_3$ of $\Gamma, \varphi(t) \Rightarrow \Delta$ with cut rank strictly below d. Now by applying first the Inversion Lemma and then the Substitution Lemma on $\mathcal{D}_0$ we obtain a derivation $\mathcal{D}_4$ of $\Gamma \Rightarrow \varphi(t), \Delta$ with cut rank strictly below d. So

$$\frac{\begin{array}{cc} \mathcal{D}_4 & \mathcal{D}_3 \\ \Gamma \;\Rightarrow\; \varphi(t), \Delta & \Gamma, \varphi(t) \;\Rightarrow\; \Delta \end{array}}{\Gamma \;\Rightarrow\; \Delta}$$

is a proof of $\Gamma \Rightarrow \Delta$ with cut rank strictly below d, as desired. □

THEOREM 1.6. (Cut elimination for the classical sequent calculus) *There is an effective method for transforming a derivation π in the classical sequent calculus for predicate logic with the cut rule into a cut free derivation π' which has the same endsequent as π.*

2. Intuitionistic sequent calculus

The axioms and rules are:

$$\text{Axioms} \quad \begin{cases} \Gamma, \varphi \Rightarrow \varphi & \text{for atomic } \varphi \\ \Gamma, \bot \Rightarrow \varphi \end{cases}$$

	Left	Right
$\wedge$	$\dfrac{\Gamma,\alpha_1,\alpha_2 \Rightarrow \varphi}{\Gamma,\alpha_1\wedge\alpha_2 \Rightarrow \varphi}$	$\dfrac{\Gamma\Rightarrow\beta_1 \quad \Gamma\Rightarrow\beta_2}{\Gamma\Rightarrow\beta_1\wedge\beta_2}$
$\vee$	$\dfrac{\Gamma,\beta_1\Rightarrow\varphi \quad \Gamma,\beta_2\Rightarrow\varphi}{\Gamma,\beta_1\vee\beta_2\Rightarrow\varphi}$	$\dfrac{\Gamma\Rightarrow\alpha_1}{\Gamma\Rightarrow\alpha_1\vee\alpha_2} \qquad \dfrac{\Gamma\Rightarrow\alpha_2}{\Gamma\Rightarrow\alpha_1\vee\alpha_2}$
$\rightarrow$	$\dfrac{\Gamma\Rightarrow\beta_1 \quad \Gamma,\beta_2\Rightarrow\varphi}{\Gamma,\beta_1\rightarrow\beta_2\Rightarrow\varphi}$	$\dfrac{\Gamma,\alpha_1\Rightarrow\alpha_2}{\Gamma\Rightarrow\alpha_1\rightarrow\alpha_2}$
$\forall$	$\dfrac{\Gamma,\varphi[t/x]\Rightarrow\psi}{\Gamma,\forall x\,\varphi\Rightarrow\psi}$	$\dfrac{\Gamma\Rightarrow\varphi}{\Gamma\Rightarrow\forall x\,\varphi}$
$\exists$	$\dfrac{\Gamma,\varphi\Rightarrow\psi}{\Gamma,\exists x\,\varphi\Rightarrow\psi}$	$\dfrac{\Gamma\Rightarrow\varphi[t/x]}{\Gamma\Rightarrow\exists x\,\varphi}$

In $\exists L$ and $\forall R$ the eigenvariable x may not occur freely in Γ or ψ.

In addition, we have the following cut rule:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi}$$

For this system we have the following lemmas:

LEMMA 2.1. (Weakening) *If $\Gamma \Rightarrow \varphi$ is the endsequent of a derivation π in the intuitionistic sequent calculus and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Rightarrow \varphi$ is derivable as well. In fact, the latter has a derivation π' with a cut rank and size no greater than that of π.*

LEMMA 2.2. (Inversion Lemma) *The following rules are invertible in the intuitionistic sequent calculus: the rules introducing* $\wedge$ *on the left and right,* $\vee$ *on the left,* $\to$ *on the right,* $\exists$ *on the left and* $\forall$ *on the right. That is, if there is a derivation* π *of a sequent* σ *and* σ *can be obtained from sequents* $\sigma_1, \ldots, \sigma_n$ *by one of these rules, then there are derivations* π_i *of the* σ_i *as well, and the cut rank of each of the* π_i *need not be any bigger than that of* π.

LEMMA 2.3. (Substitution Lemma) *If* π *is a derivation of* $\Gamma \Rightarrow \varphi$*, then there is a derivation of* $\Gamma[t/x] \Rightarrow \varphi[t/x]$ *with a derivation which has no greater cut rank or size than* π.

THEOREM 2.4. (Cut elimination for the intuitionistic sequent calculus) *There is an effective method for transforming a derivation* π *in the intuitionistic sequent calculus for predicate logic into a cut free derivation* π' *which has the same endsequent as* π.

3. Applications

Cut free derivations still obey a form of the subformula property.

DEFINITION 3.1. The collection $\mathrm{GSub}(\varphi)$ of *Gentzen subformulas* of φ is defined by induction on the structure of φ as follows:

$$\begin{array}{rcl} \mathrm{GSub}(\varphi) & = & \{\varphi\} \quad \text{if } \varphi \text{ is atomic} \\ \mathrm{GSub}(\varphi \Box \psi) & = & \mathrm{GSub}(\varphi) \cup \mathrm{GSub}(\psi) \cup \{\varphi \Box \psi\} \\ \mathrm{GSub}(Qx\,\varphi) & = & \{Qx\,\varphi\} \cup \bigcup \{\, \mathrm{GSub}(\varphi[t/x]) : t \text{ term}\} \end{array}$$

with $\Box \in \{\wedge, \vee, \to\}$ and $Q \in \{\forall, \exists\}$.

LEMMA 3.2. *A cut free derivation* π *of a sequent* σ *in either the classical or intuitionistic sequent calculus only contains Gentzen subformulas of formulas occurring in* σ.

But note that the definition of Gentzen subformula is such that $\forall x\, Px$ has infinitely many Gentzen subformulas: indeed, each formula of the form $P(t)$ is a Gentzen subformula. For that reason we cannot use the subformula property in combination with cut elimination to argue for the decidability of either classical or intuitionistic predicate logic. Which is just as well, because both are in fact undecidable.

For intuitionistic logic we obtain an effective proof of the existence property.

THEOREM 3.3. *If the sequent* $\Rightarrow \exists x\, \varphi$ *is derivable in the intuitionistic sequent calculus, then there is a term* t *such that* $\Rightarrow \varphi[t/x]$ *is derivable as well.*

PROOF. Indeed, the last step in a cut free derivation of $\Rightarrow \exists x\, \varphi$ in the intuitionistic sequent calculus must have been the rule introducing an existential quantifier on the right. For this reason it contains a derivation of $\Rightarrow \varphi(t)$ for some term t. □

For classical predicate logic we have a version of this as well.

THEOREM 3.4. (Herbrand's Theorem) *If the sequent* $\Rightarrow \exists x\, \varphi$ *is derivable in the classical sequent calculus and* φ *is quantifier-free, then there are terms* $t_1, \ldots, t_n$ *such that* $\Rightarrow \varphi(t_1) \vee \varphi(t_2) \vee \ldots \vee \varphi(t_n)$ *is derivable as well.*

PROOF. Use induction on derivations to prove that if $\Gamma \Rightarrow \Delta, \exists x\, \varphi$ is derivable and Γ, Δ and φ are quantifier-free, then there are terms $t_1, \ldots, t_n$ such that $\Gamma \Rightarrow \Delta, \varphi(t_1), \ldots, \varphi(t_n)$ is derivable as well. □

Initially, the case considered in the previous theorem may look rather special. However, in classical logic any formula φ can be brought in prenex normal form, that is, it can be rewritten as:

$$\exists x_0 \, \forall y_0 \, \exists x_1 \, \forall y_1 \ldots \exists x_n \forall y_n \psi(x_0, \ldots, x_n, y_0, \ldots, y_n),$$

with ψ quantifier-free. And a formula of this form is a tautology if and only if

$$\exists x_0 \ldots \exists x_n \, \psi(x_0, \ldots, x_n, f_0(x_0), f_1(x_0, x_1), \ldots, f_n(x_0, \ldots, x_n))$$

is a tautology, provided the function symbols $f_0, \ldots, f_n$ did not already occur in ψ (such function symbols are called *Herbrand functions*, which are "dual" to the more familiar Skolem functions). This observation in combination with Herbrand's Theorem is heavily exploited in logic programming.

But this observation also means that the question of deciding which first-order formulas are tautologies is as difficult as deciding which existential formulas are tautologies. So the problem of deciding first-order logic can be reduced completely to the problem of finding the correct Herbrand terms. Which means, in particular, that this problem must be undecidable (so there can be no algorithm which given an arbitrary existential sentence outputs a list of terms which would be a correct list of Herbrand terms in case the existential formula is a tautology).