CHAPTER 7

Cut elimination

We have seen that both the classical and the intuitionistic sequent calculus are already complete without the cut rule. That means that whenever there is a derivation π in the sequent calculus with the cut rule there must also be a derivation π' with the same endsequent in the sequent calculus without the cut rule. What such a completeness proof does not give us is a method for effectively transforming such a derivation π into a cut free proof π'. In this chapter we will outline such an effective procedure for *cut elimination*, as it is called. We will see it can be used to give effective proofs of some other properties as well.

1. Cut elimination for the classical sequent calculus

We work in the classical sequent calculus with the cut rule.

First a definition:

DEFINITION 1.1. The logical depth $\mathrm{dp}(\varphi)$ of a formula is defined inductively as follows: the logical depth of a propositional variable or $\perp$ is 0, while the logical depth of $\varphi \Box \psi$ is $\max(\mathrm{dp}(\varphi), \mathrm{dp}(\psi)) + 1$. The rank $\mathrm{rk}(\varphi)$ of a formula φ will be defined as $\mathrm{dp}(\varphi) + 1$.

If π is a derivation, then we define its *cut rank* to be 0, if it contains no cut formulas (i.e., is cut free). If, on the other hand, it contains applications of the cut rule, then we define the cut rank of π to be the maximum of all the ranks of cut formulas in π.

Before we proceed to eliminate cuts, we first need to strengthen the weakening and inversion lemmas.

LEMMA 1.2. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a derivation π and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is derivable as well. In fact, the latter has a derivation π' with a cut rank no greater than that of π.*

LEMMA 1.3. (Inversion Lemma) *Each of the rules in the classical sequent calculus is invertible: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one the rules, then there are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π.*

The key step in the proof for cut elimination is the following:

LEMMA 1.4. (Key Lemma) *Suppose π is a derivation which ends with an application of the cut rule applied to a formula of rank d, while the rank of any other cut formula in π is strictly smaller than d. Then π can be transformed into a derivation π' with the same endsequent as π and with cut rank strictly less than d.*

PROOF. The idea is to look at the structure of the cut formula of rank d. Suppose it is of the form $\varphi \wedge \psi$, say, so that the last step in the proof looks like this:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \;\Rightarrow\; \varphi\wedge\psi,\Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma,\varphi\wedge\psi \;\Rightarrow\; \Delta\end{matrix}}{\Gamma \;\Rightarrow\; \Delta}$$

The inversion lemma says that we may assume, without loss of generality, that the last rules that were applied in the $\mathcal{D}_i$ were the $\wedge$-introduction rules introducing $\varphi \wedge \psi$, so that the derivation looks like this:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00}\\ \Gamma \;\Rightarrow\; \varphi,\Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_{01}\\ \Gamma \;\Rightarrow\; \psi,\Delta\end{matrix}}{\Gamma \;\Rightarrow\; \varphi\wedge\psi,\Delta} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma,\varphi,\psi \;\Rightarrow\; \Delta\end{matrix}}{\Gamma,\varphi\wedge\psi \;\Rightarrow\; \Delta}}{\Gamma \;\Rightarrow\; \Delta}$$

In this case we replace this derivation by:

$$\dfrac{\begin{matrix}\mathcal{D}_{01}\\ \Gamma \;\Rightarrow\; \psi,\Delta\end{matrix} \qquad \dfrac{\begin{matrix}\mathcal{D}'_{00}\\ \Gamma,\psi \;\Rightarrow\; \varphi,\Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma,\varphi,\psi \;\Rightarrow\; \Delta\end{matrix}}{\Gamma,\psi \;\Rightarrow\; \Delta}}{\Gamma \;\Rightarrow\; \Delta}$$

where $\mathcal{D}'_{00}$ is obtained from $\mathcal{D}_{00}$ by using weakening. In this derivation there is now one more application of the cut rule but the cut rank is now strictly less than d, which is what we wanted.

The cases for disjunction and implication are similar, so it remains to consider the case where the cut formula is of the form p, for a propositional variable p:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \;\Rightarrow\; p,\Delta\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma,p \;\Rightarrow\; \Delta\end{matrix}}{\Gamma \;\Rightarrow\; \Delta}$$

By assumption $\mathcal{D}_0$ and $\mathcal{D}_1$ are cut free. In this case we perform the following somewhat delicate operation on the derivation $\mathcal{D}_1$: first of all, we add everywhere Γ to the left and Δ to the right of the arrow $\Rightarrow$. Then we are going to delete some *passive* occurrences of p on the *left* of the arrow: we start from the bottom of the tree and delete p on the left of the endsequent. Then we climb up in the tree and delete ps on the left as long as they are passive. As soon as we see an active occurrence of p on the left, we stop and leave all the passive occurrences above this active occurrence alone. The result of this operation, which we may call $\mathcal{D}'_1$, need no longer be a correct derivation. However, the key observation is that the only way in which $\mathcal{D}'_1$ could fail to be a correct derivation is that $\mathcal{D}_1$ may have contained axioms of the form $\Gamma', p \Rightarrow \Delta', p$ which are no longer axioms in $\mathcal{D}'_1$, because the p on the left has disappeared, so that in $\mathcal{D}'_1$ we just see $\Gamma, \Gamma' \Rightarrow p, \Delta, \Delta'$ (remember: we have added Γ and Δ everywhere). In that case we apply weakening to $\mathcal{D}_0$ to regard it as a derivation of $\Gamma, \Gamma' \Rightarrow p, \Delta, \Delta'$ and stick this derivation onto the derivation $\mathcal{D}'_1$. The result is a cut free derivation of $\Gamma \Rightarrow \Delta$. □

REMARK 1.5. The last step in the proof is perfectly symmetrical in $\mathcal{D}_0$ and $\mathcal{D}_1$, so we could also have added Γ and Δ everywhere in $\mathcal{D}_0$ and then started deleting passive occurrences of p on the *right* in $\mathcal{D}_0$ in the manner just described. What is most convenient depends on the precise shape of the proof. However, when we turn to the intuitionistic case, the symmetry is broken and we really have to do what we just did in the proof above and work on the derivation $\mathcal{D}_1$.

THEOREM 1.6. (Cut elimination for classical propositional logic) *There is an effective method for transforming a derivation π in the classical sequent calculus with the cut rule into a cut free derivation π' which has the same endsequent as π.*

PROOF. Suppose d is the cut rank of π, so that there are some cut formulas with rank d in the tree, but no cut formula with higher rank. The idea is to replace these cut formulas with cut formulas of lower rank, starting with those that are highest up in the tree (meaning that there are no cut formulas of rank d or higher above them), as these can be eliminated using Lemma 1.4. So by repeated application of this lemma we eliminate all the cut formula of rank d, bringing down the cut rank of π. By repeated application of this procedure we will ultimately bring down the cut rank to 0, meaning that the proof is cut free. □

2. Cut elimination for the intuitionistic sequent calculus

We try to use the same ideas as in the previous section to give a cut elimination proof for the intuitionistic sequent calculus. The main difficulty we face is that the Inversion Lemma fails in general. However, for the same notion of cut rank we do still have:

LEMMA 2.1. (Weakening) *If $\Gamma \Rightarrow \varphi$ is the endsequent of a derivation π in the intuitionistic sequent calculus and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Rightarrow \varphi$ is derivable as well. In fact, the latter has a derivation π' with a cut rank and size no greater than that of π.*

LEMMA 2.2. (Inversion Lemma) *The rules for introducing conjunctions on the left and right, implications on the right and disjunctions on the left are invertible in the intuitionistic sequent calculus: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one of these rules, then there are derivations π_i of the σ_i as well, and the cut rank of each of the π_i need not be any bigger than that of π.*

LEMMA 2.3. (Key Lemma) *Suppose π is a derivation which ends with an application of the cut rule applied to a formula of rank d, while the rank of any other application of the cut rule in π is strictly smaller than d. Then π can be transformed into a derivation π' with the same endsequent as π and which has cut rank strictly less than d.*

PROOF. We try to mimick the proof in the classical case, but, of course, we have to take into account the fact that not every rule in the intuitionistic sequent calculus is invertible; what saves us, however, is that for each connective there is one introduction rule, either on the left or right, which is invertible.

We now prove the Key Lemma by induction on the size of π.

The case for $d = 1$ is the same as for the classical sequent calculus (see also Remark 1.5), so suppose $d > 1$. In case the cut formula of rank d is of the form $\varphi \wedge \psi$, we can argue as in the classical case as well, so the interesting cases are where this cut formula is an implication $\varphi \to \psi$ or a disjunction $\varphi \vee \psi$.

Let us first consider the case where the cut formula of rank d is of the form $\varphi \to \psi$, so our derivation π has the following shape:

$$\dfrac{\begin{array}{c}\mathcal{D}_0 \\ \Gamma \Rightarrow \varphi \to \psi\end{array} \qquad \begin{array}{c}\mathcal{D}_1 \\ \Gamma, \varphi \to \psi \Rightarrow \chi\end{array}}{\Gamma \Rightarrow \chi}$$

We now make a case distinction on what was the last rule which was applied in $\mathcal{D}_1$; of course, it could that $\Gamma, \varphi \to \psi \Rightarrow \chi$ is an axiom, but then $\Gamma \Rightarrow \chi$ is as well and we are finished.

Another possibility is that the final inference step in $\mathcal{D}_1$ was an introduction rule in which $\varphi \to \psi$ is a side formula, meaning that it introduces some formula different from $\varphi \to \psi$ on the left or right, say $\alpha \wedge \beta$ on the right:

$$\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \varphi \to \psi} \qquad \dfrac{\dfrac{\mathcal{D}_{10}}{\Gamma, \varphi \to \psi \Rightarrow \alpha} \quad \dfrac{\mathcal{D}_{11}}{\Gamma, \varphi \to \psi \Rightarrow \beta}}{\Gamma, \varphi \to \psi \Rightarrow \alpha \wedge \beta}}{\Gamma \Rightarrow \alpha \wedge \beta}$$

with $\chi = \alpha \wedge \beta$. In this case we apply the induction hypothesis to

$$\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \varphi \to \psi} \qquad \dfrac{\mathcal{D}_{10}}{\Gamma, \varphi \to \psi \Rightarrow \alpha}}{\Gamma \Rightarrow \alpha}$$

as well as to

$$\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \varphi \to \psi} \qquad \dfrac{\mathcal{D}_{11}}{\Gamma, \varphi \to \psi \Rightarrow \beta}}{\Gamma \Rightarrow \beta}$$

and we obtain derivations of $\Gamma \Rightarrow \alpha$ and $\Gamma \Rightarrow \beta$ with cut rank strictly below d. By applying $\wedge$-introduction on the right to these we obtain a proof of $\Gamma \Rightarrow \alpha \wedge \beta$ with cut rank strictly below d.

Another possibility is that the final rule which was applied in $\mathcal{D}_1$ was the cut rule applied to a formula θ with rank strictly less than d:

$$\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \varphi \to \psi} \qquad \dfrac{\dfrac{\mathcal{D}_{10}}{\Gamma, \varphi \to \psi \Rightarrow \theta} \quad \dfrac{\mathcal{D}_{11}}{\Gamma, \varphi \to \psi, \theta \Rightarrow \chi}}{\Gamma, \varphi \to \psi \Rightarrow \chi}}{\Gamma \Rightarrow \chi}$$

In this case we apply the induction hypothesis to

$$\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \varphi \to \psi} \qquad \dfrac{\mathcal{D}_{10}}{\Gamma, \varphi \to \psi \Rightarrow \theta}}{\Gamma \Rightarrow \theta}$$

as well as to

$$\dfrac{\dfrac{\mathcal{D}_0'}{\Gamma, \theta \Rightarrow \varphi \to \psi} \qquad \dfrac{\mathcal{D}_{11}}{\Gamma, \varphi \to \psi, \theta \Rightarrow \chi}}{\Gamma, \theta \Rightarrow \chi}$$

where we have weakened $\mathcal{D}_0$ by adding θ on the left, resulting in two derivations $\mathcal{D}_2$ and $\mathcal{D}_3$ of $\Gamma \Rightarrow \theta$ and $\Gamma, \theta \Rightarrow \chi$, respectively, both with cut rank strictly below d. But then

$$\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow \theta\end{array} \qquad \begin{array}{c}\mathcal{D}_3\\ \Gamma, \theta \Rightarrow \chi\end{array}}{\Gamma \Rightarrow \chi}$$

is a derivation of $\Gamma \Rightarrow \chi$ with cut rank strictly less than d.

The only other possibility is that the final rule in $\mathcal{D}_1$ introduced $\varphi \to \psi$, which means that π either looks like this

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \varphi \to \psi\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{10}\\ \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_{11}\\ \Gamma, \psi \Rightarrow \chi\end{array}}{\Gamma, \varphi \to \psi \Rightarrow \chi}}{\Gamma \Rightarrow \chi}$$

or like this

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \varphi \to \psi\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{10}\\ \Gamma, \varphi \to \psi \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_{11}\\ \Gamma, \varphi \to \psi, \psi \Rightarrow \chi\end{array}}{\Gamma, \varphi \to \psi \Rightarrow \chi}}{\Gamma \Rightarrow \chi}$$

Clearly, if we have a derivation of the first type we can also obtain a derivation of the second type by weakening, so we only consider the second possibility. In that case we apply the induction hypothesis to

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \varphi \to \psi\end{array} \qquad \begin{array}{c}\mathcal{D}_{10}\\ \Gamma, \varphi \to \psi \Rightarrow \varphi\end{array}}{\Gamma \Rightarrow \varphi}$$

and to

$$\dfrac{\begin{array}{c}\mathcal{D}_0'\\ \Gamma, \psi \Rightarrow \varphi \to \psi\end{array} \qquad \begin{array}{c}\mathcal{D}_{11}\\ \Gamma, \varphi \to \psi, \psi \Rightarrow \chi\end{array}}{\Gamma, \psi \Rightarrow \chi}$$

to obtain derivations $\mathcal{D}_2$ and $\mathcal{D}_3$ of $\Gamma \Rightarrow \varphi$ and $\Gamma, \psi \Rightarrow \chi$, respectively, with cut rank strictly below d. Now, using the Inversion Lemma on $\mathcal{D}_0$ we also have a derivation $\mathcal{D}_4$ of $\Gamma, \varphi \Rightarrow \psi$ with cut rank below d, so

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_2\\ \Gamma \Rightarrow \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_4\\ \Gamma, \varphi \Rightarrow \psi\end{array}}{\Gamma \Rightarrow \psi} \qquad \begin{array}{c}\mathcal{D}_3\\ \Gamma, \psi \Rightarrow \chi\end{array}}{\Gamma \Rightarrow \chi}$$

is a proof of $\Gamma \Rightarrow \Delta$ with cut rank strictly below d, as desired.

The case where the unique cut formula of rank d is a disjunction $\varphi \vee \psi$ is similar, but easier and therefore omitted. (In fact, at this point this would make for a nice exercise!) □

As before we now have:

THEOREM 2.4. (Cut elimination for the intuitionistic sequent calculus) *There is an effective method for transforming a derivation* π *in the intuitionistic sequent calculus with the cut rule into a cut free derivation* π' *which has the same endsequent as* π.

As a result we also obtain constructive proofs of the following theorems:

THEOREM 2.5. (Decidability of intuitionistic propositional logic) *The question whether a propositional formula* φ *is derivable in intuitionistic natural deduction (or whether an intuitionistic sequent* σ *is derivable in the sequent calculus for intuitionistic propositional logic) is decidable.*

PROOF. A formula φ is derivable in intuitionistic natural deduction iff $\Rightarrow \varphi$ is derivable in the intuitionistic sequent calculus iff $\Rightarrow \varphi$ has a cut free proof. But we have seen already that the question whether a sequent has cut free derivation in the sequent calculus for intuitionistic propositional logic is decidable. □

THEOREM 2.6. (Disjunction property for intuitionistic propositional logic) *If* $\Rightarrow \varphi \vee \psi$ *is provable in the intuitionistic sequent calculus, then so is either* $\Rightarrow \varphi$ *or* $\Rightarrow \psi$.

PROOF. Any derivation π of $\Rightarrow \varphi \vee \psi$ can be transformed into a cut free derivation π' with the same sequent $\Rightarrow \varphi \vee \psi$. But the last inference step in such a cut free derivation can only have been the rule introducing $\vee$ on the right, so a cut free proof $\Rightarrow \varphi \vee \psi$ can only have been obtained from a proof of $\Rightarrow \varphi$ or $\Rightarrow \psi$. □