CHAPTER 16

Term extraction

One would expect that implicit in a constructive proof of an existential statement like $\exists x^\sigma\, \varphi(x)$ would be a method for finding a term t such that $\varphi(t)$ holds. Below we will show that this is correct for HA^ω. More generally, we will show that one can extract from a proof of

$$\forall x^\sigma\, \exists y^\tau\, \varphi(x, y)$$

a term t such that $\forall x^\sigma\, \varphi(x, tx)$ holds. In order to show this we need to modify modified realizability and that is what we will do in the first section.

In the second section we will show that similar results hold for PA^ω as well, provided φ is simple enough.

1. Modified realizability with truth

In order to show our term extract results we need to modify modified realizability in such a way that HA^ω believes that anything which is realized is also true. We have seen that for ordinary modified realizability this is not the case, because AC is both unprovable and provably mr-realized in HA^ω. To get what we want, we only have to make a slight change to the definition of mr: indeed, the notion of a type $\mathrm{tp}(\varphi)$ of a formula φ is as before and we only add a clause saying that $\varphi \to \psi$ can only be realized if it is true. The result is called "modified realizability with truth" and is defined as follows:

Definition 1.1. To any formula φ in the language of HA^ω we associate a new formula $x \;\mathsf{mrt}\; \varphi$ as follows, where $x \;\mathsf{mrt}\; \varphi$ is a formula in the language of HA^ω whose free variables are those of φ plus possibly a variable x of type $\mathrm{tp}(\varphi)$:

$$\begin{aligned}
x \;\mathsf{mrt}\; \varphi &:= \varphi \qquad \text{if } \varphi \text{ is atomic.} \\
x \;\mathsf{mrt}\, (\varphi \wedge \psi) &:= \mathbf{p}_0 x \;\mathsf{mrt}\; \varphi \wedge \mathbf{p}_1 x \;\mathsf{mrt}\; \psi \\
x \;\mathsf{mrt}\, (\varphi \to \psi) &:= (\varphi \to \psi) \wedge \forall y^{\mathrm{tp}(\varphi)}\, (\, y \;\mathsf{mrt}\; \varphi \to x(y) \;\mathsf{mrt}\; \psi\,) \\
x \;\mathsf{mrt}\; \exists y^\sigma\, \varphi &:= \mathbf{p}_1 x \;\mathsf{mrt}\; \varphi(\mathbf{p}_0 x) \\
x \;\mathsf{mrt}\; \forall y^\sigma\, \varphi &:= \forall y^\sigma\, (\, x(y) \;\mathsf{mrt}\; \varphi\,)
\end{aligned}$$

Proposition 1.2. *For any formula φ in the language of HA^ω we have*

$$\mathsf{HA}^\omega \vdash (x \;\mathsf{mrt}\, \varphi) \to \varphi.$$

Proof. By induction on the structure of φ. □

Theorem 1.3. *Let φ be a formula in the language of HA^ω. If φ is provable in HA^ω, then one can find effectively from this proof a term t in the language of HA^ω such that:*

1. *any variables occurring freely in t also occur freely in φ, and*
2. $\mathsf{HA}^\omega \vdash t \;\mathsf{mrt}\; \varphi$.

The same statement holds for $\mathsf{E\text{-}HA}^\omega$.

PROOF. As for modified realizability. □

COROLLARY 1.4. (Term extraction for HA^ω) *Consider a formula of the form* $\forall x^\sigma\, \exists y^\tau\, \varphi(x,y)$. *If this formula is provable in* HA^ω, *then from this proof one can effectively extract a term* t *of type* $\sigma \to \tau$ *such that*

$$\mathsf{HA}^\omega \vdash \forall x^\sigma\, \varphi(x, tx),$$

where the variables occurring freely in t *also occur freely in* $\forall x^\sigma\, \exists y^\tau\, \varphi(x,y)$. *In particular, if a formula of the form* $\exists x^\sigma\, \varphi(x)$ *is provable in* HA^ω, *then one can find a term* t *of type* σ *such that* $\varphi(t)$ *is provable in* HA^ω *as well. The same statements hold for* $\mathsf{E\text{-}HA}^\omega$ *as well.*

PROOF. If $\forall x^\sigma\, \exists y^\tau\, \varphi(x,y)$ is provable, then it follows from the soundness of modified realizability with truth that there is a term s such that

$$\mathsf{HA}^\omega \vdash s \;\mathsf{mrt}\; \forall x^\sigma\, \exists y^\tau\, \varphi(x,y).$$

By definition of mrt this means that

$$\mathsf{HA}^\omega \vdash \forall x^\sigma\, \mathbf{p}_1(sx) \;\mathsf{mrt}\; \varphi(x, \mathbf{p}_0(sx)).$$

So if we put $t = \lambda x^\sigma.\mathbf{p}_0(sx)$, then

$$\mathsf{HA}^\omega \vdash \forall x^\sigma\, \mathbf{p}_1(sx) \;\mathsf{mrt}\; \varphi(x, tx).$$

But then Proposition 1.2 implies that

$$\mathsf{HA}^\omega \vdash \forall x^\sigma\, \varphi(x, tx),$$

as desired. □

COROLLARY 1.5. (Numerical existence property for HA^ω) *If a sentence of the form* $\exists x^0\, \varphi(x)$ *is provable in* HA^ω, *then there is a numeral* $S^n 0$ *such that* $\varphi(S^n 0)$ *is provable in* HA^ω *as well. The same statement holds for* $\mathsf{E\text{-}HA}^\omega$ *as well.*

PROOF. Suppose $\exists x^0\, \varphi(x)$ is provable in HA^ω and x is the only variable occurring freely in $\varphi(x)$. Then the previous corollary tells us that there is a closed term t of type 0 such that HA^ω proves $\varphi(t)$. By strong normalisation for Gödel's $\mathcal{T}$ the term t has a normal form, which will be some numeral $S^n 0$. Then $\mathsf{HA}^\omega \vdash t = S^n 0$ and $\mathsf{HA}^\omega \vdash \varphi(S^n 0)$. □

COROLLARY 1.6. (Disjunction property for HA^ω) *If a sentence of the form* $\varphi \vee \psi$ *is provable in* HA^ω, *then either* φ *or* ψ *is provable in* HA^ω. *The same statement holds for* $\mathsf{E\text{-}HA}^\omega$ *as well.*

PROOF. Remember that we treat $\varphi \vee \psi$ as an abbreviation of

$$\exists n^0\, \big((n = 0 \to \varphi) \wedge (n \neq 0 \to \psi) \big).$$

So this follows from the previous corollary. □

2. Classical arithmetic

In this section we will show a term extraction result for classical arithmetic. This will be a consequence of the combination of two things: term extraction for HA^ω and the fact PA^ω is Π^0_2-conservative over HA^ω (that is, PA^ω proves the same Π^0_2-formulas as HA^ω). Since we already proved the former, we only have to prove the latter. But before we can do that, we first have to define the notion of a Π^0_2-formula and prove two lemmas.

DEFINITION 2.1. We will call a formula φ *simple* if it contains neither quantifiers nor equalities $=_\sigma$ with $\sigma \neq 0$ (this is not standard terminology). A formula of the form

$$\forall x^\sigma \, \exists y^\tau \, \varphi$$

where φ is simple is called a Π^0_2*-formula*.

LEMMA 2.2. *For any simple formula φ with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ there is a closed term $\mathbf{d}$ of type $\sigma \to \ldots \to \sigma_n \to 0$ such that*

$$\mathsf{HA}^\omega \vdash \forall x_1^{\sigma_1}, \ldots, \forall x_n^{\sigma_n} \big(\varphi \leftrightarrow \mathbf{d} x_1 \ldots x_n =_0 0 \big).$$

PROOF. Exercise! □

LEMMA 2.3. *Let ∇ be a function mapping formulas in the language of HA^ω to formulas in the language in HA^ω, such that the following statements are provable in HA^ω:*

$$\begin{array}{l}
\mathsf{HA}^\omega \vdash \varphi \to \nabla\varphi \\
\mathsf{HA}^\omega \vdash \nabla(\varphi \wedge \psi) \leftrightarrow (\, \nabla\varphi \wedge \nabla\psi \,) \\
\mathsf{HA}^\omega \vdash (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi) \\
\mathsf{HA}^\omega \vdash (\nabla\varphi)[t/x] \leftrightarrow \nabla(\varphi[t/x])
\end{array}$$

In addition, let φ^∇ be the formula obtained from φ by applying ∇ to each atomic subformula, disjunction and existentially quantified subformula. More precisely, φ^∇ is defined by induction on the structure of φ as follows:

$$\begin{array}{rcll}
\varphi^\nabla & := & \nabla\varphi & \textit{if } \varphi \textit{ is an atomic formula}, \\
(\varphi \wedge \psi)^\nabla & := & \varphi^\nabla \wedge \psi^\nabla, & \\
(\varphi \vee \psi)^\nabla & := & \nabla(\varphi^\nabla \vee \psi^\nabla), & \\
(\varphi \to \psi)^\nabla & := & \varphi^\nabla \to \psi^\nabla, & \\
(\forall x \, \varphi(x)\,)^\nabla & := & \forall x \, (\varphi(x))^\nabla, & \\
(\exists x \, \varphi(x)\,)^\nabla & := & \nabla \exists x \, (\varphi(x))^\nabla. &
\end{array}$$

Then $\mathsf{HA}^\omega \vdash \varphi$ implies $\mathsf{HA}^\omega \vdash \varphi^\nabla$.

PROOF. Since we already know this for predicate logic, it suffices to check that φ^∇ is provable in HA^ω if φ is a non-logical axiom of HA^ω. We leave this as an exercise. □

In particular, we have that if φ is provable in PA^ω then $\varphi^{\neg\neg}$ (its double negation translation) is provable in HA^ω. But we also have the following:

THEOREM 2.4. *Any Π^0_2-formula provable in PA^ω is also provable in HA^ω.*

Proof. Let φ be a Π^0_2-formula. In view of Lemma 2.2 we may assume that φ is of the form

$$\forall x^\sigma \, \exists y^\tau \, t(x, y) =_0 0.$$

We know that for any formula A the nucleus $\nabla\varphi := (\varphi \to A) \to A$ interprets classical logic. This means that if φ is provable in PA^ω, then

$$\mathsf{HA}^\omega \vdash \forall x^\sigma \, \nabla(\exists y^\tau \, \nabla(t(x, y) = 0)).$$

In other words, we have

$$\mathsf{HA}^\omega \vdash \forall x^\sigma \, \big(([\exists y^\tau \, ((t(x, y) = 0 \to A) \to A)] \to A) \to A \big)$$

for any formula A. But if we choose A to be $\exists y^\tau \, t(x, y) = 0$, then this is equivalent to

$$\mathsf{HA}^\omega \vdash \forall x^\sigma \, \exists y^\tau \, t(x, y) = 0.$$

So φ is provable in HA^ω. □

The proof method that we employed above is often called Friedman's trick (and is due to Harvey Friedman).

Corollary 2.5. (Term extraction for PA^ω) *Consider a formula of the form* $\forall x^\sigma \, \exists y^\tau \, \varphi(x, y)$, *where* φ *is simple. If this formula is provable in* PA^ω, *then from this proof one can effectively extract a term* t *of type* $\sigma \to \tau$ *such that*

$$\mathsf{PA}^\omega \vdash \forall x^\sigma \, \varphi(x, tx).$$

In particular, if a formula of the form $\exists x^\sigma \, \varphi(x)$ *with* φ *simple is provable in* PA^ω, *then one can find a term* t *of type* σ *such that* $\varphi(t)$ *is provable in* PA^ω *as well.*

Proof. Follows from Theorem 2.4 and Corollary 1.4. □