# 8th Exercise sheet Proof Theory
# 7 Mar 2018

**Exercise 1** Consider the following De Morgan laws:

$$\neg\exists x\,\varphi \to \forall x\,\neg\varphi$$
$$\forall x\,\neg\varphi \to \neg\exists x\,\varphi$$
$$\neg\forall x\,\varphi \to \exists x\,\neg\varphi$$
$$\exists x\,\neg\varphi \to \neg\forall x\,\varphi$$

Which ones of those are also intuitionistically valid? Give an intuitionistic natural deduction-style proof, if possible; if this is not possible, give a classical proof and a Kripke model refuting the statement.

**Exercise 2** Consider the following classical tautologies:

$$(\forall x\,\varphi \to \psi) \to \exists x\,(\varphi \to \psi)$$
$$(\exists x\,\varphi \to \psi) \to \forall x\,(\varphi \to \psi)$$
$$(\psi \to \forall x\,\varphi) \to \forall x(\psi \to \varphi)$$
$$(\psi \to \exists x\,\varphi) \to \exists x(\psi \to \varphi)$$

(Here $\psi$ is a formula in which $x$ does not occur freely.) Which ones of those are also intuitionistically valid? Give an intuitionistic natural deduction-style proof, if possible; if this is not possible, give a classical proof and a Kripke model refuting the statement.

**Exercise 3** Construct a Kripke model refuting the intuitionistic validity of the sentence

$$\neg\neg\forall x\,(\,A(x) \vee \neg A(x)\,).$$

This shows that there are formulas $\varphi$ in predicate logic such that $\varphi$ is a classical tautology, while not even $\neg\neg\varphi$ is an intuitionistic tautology.

**Exercise 4** We extend the theory of nuclei to predicate logic. So now a nucleus is a function $\nabla$ sending formulas in predicate logic to formulas in predicate logic, in such a way that the following statements are provable in intuitionistic logic:

$$\begin{array}{l} \vdash_{\mathrm{IL}} \varphi \to \nabla\varphi \\ \vdash_{\mathrm{IL}} \nabla(\varphi \wedge \psi) \leftrightarrow (\, \nabla\varphi \wedge \nabla\psi \,) \\ \vdash_{\mathrm{IL}} (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi) \end{array}$$

In addition, define $\varphi^{\nabla}$ by induction on the structure of $\varphi$ as follows:

$$\begin{array}{rcll} \varphi^{\nabla} & := & \nabla\varphi & \text{if } \varphi \text{ is a propositional variable or } \bot, \\ (\varphi \wedge \psi)^{\nabla} & := & \varphi^{\nabla} \wedge \psi^{\nabla}, & \\ (\varphi \vee \psi)^{\nabla} & := & \nabla(\varphi^{\nabla} \vee \psi^{\nabla}), & \\ (\varphi \to \psi)^{\nabla} & := & \varphi^{\nabla} \to \psi^{\nabla}, & \\ (\forall x\, \varphi(x)\,)^{\nabla} & := & \forall x\, (\varphi(x))^{\nabla}, & \\ (\exists x\, \varphi(x)\,)^{\nabla} & := & \nabla\exists x\, (\varphi(x))^{\nabla}. & \end{array}$$

(a) Show $\vdash_{\mathrm{IL}} \nabla\exists x\, \nabla\varphi \leftrightarrow \nabla\exists x\, \varphi$ and $\vdash_{\mathrm{IL}} \nabla\forall x\, \nabla\varphi \leftrightarrow \forall x\, \nabla\varphi$

(b) Show that for any formula $\varphi$ we have $\vdash_{\mathrm{IL}} \nabla\varphi^{\nabla} \leftrightarrow \varphi^{\nabla}$.

(c) Show that $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{IL}} \psi$ implies $\varphi_1^{\nabla}, \ldots, \varphi_n^{\nabla} \vdash_{\mathrm{IL}} \psi^{\nabla}$.