3rd Exercise sheet Proof Theory
16 Feb 2018

Recall that a nucleus is a function mapping formulas in propositional logic to formulas in propositional logic for which the following statements are provable in intuitionistic logic:

$$\vdash_{\mathrm{IL}} \varphi \to \nabla\varphi$$
$$\vdash_{\mathrm{IL}} \nabla(\varphi \wedge \psi) \leftrightarrow (\nabla\varphi \wedge \nabla\psi)$$
$$\vdash_{\mathrm{IL}} (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi)$$

Exercise 1 Let A be some fixed propositional formula. Check that $\nabla\varphi := \varphi \vee A$ and $\nabla\varphi := (\varphi \to A) \to A$ are nuclei.

Exercise 2 Show that for any nucleus ∇ we have:

$$\vdash_{\mathrm{IL}} (\varphi \to \psi) \to (\nabla\varphi \to \nabla\psi)$$
$$\vdash_{\mathrm{IL}} \nabla\varphi \leftrightarrow \nabla\nabla\varphi$$
$$\vdash_{\mathrm{IL}} \nabla(\nabla\varphi \vee \nabla\psi) \leftrightarrow \nabla(\varphi \vee \psi)$$
$$\vdash_{\mathrm{IL}} \nabla(\varphi \to \nabla\psi) \leftrightarrow (\nabla\varphi \to \nabla\psi)$$

Given a nucleus ∇, let φ^∇ be the formula obtained from φ by applying ∇ to each propositional variable and each disjunction. More precisely, let φ^∇ be defined by induction on the structure of φ as follows:

$$\begin{array}{rcll} \varphi^\nabla & := & \nabla\varphi & \text{if } \varphi \text{ is a propositional variable or } \bot, \\ (\varphi \wedge \psi)^\nabla & := & \varphi^\nabla \wedge \psi^\nabla, & \\ (\varphi \vee \psi)^\nabla & := & \nabla(\varphi^\nabla \vee \psi^\nabla), & \\ (\varphi \to \psi)^\nabla & := & \varphi^\nabla \to \psi^\nabla. & \end{array}$$

Exercise 3 (a) Show that for any formula φ we have $\vdash_{\mathrm{IL}} \nabla\varphi^\nabla \leftrightarrow \varphi^\nabla$.

(b) Show that $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{IL}} \psi$ implies $\varphi_1^\nabla, \ldots, \varphi_n^\nabla \vdash_{\mathrm{IL}} \psi^\nabla$.

Exercise 4 In this exercise we will only consider nuclei of the form $\nabla\varphi := (\varphi \to A) \to A$ for some fixed propositional formula A.

(a) Show $\vdash_{\mathrm{IL}} \nabla\bot \leftrightarrow A$.

(b) Show that $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{CL}} \psi$ implies $\varphi_1^\nabla, \ldots, \varphi_n^\nabla \vdash_{\mathrm{IL}} \psi^\nabla$.

(c) Show that if $A = \bot$, that is, if $\nabla\varphi = \neg\neg\varphi$, then $\vdash_{\mathrm{CL}} \varphi \leftrightarrow \varphi^\nabla$.