11th Exercise sheet Proof Theory
16 Mar 2018

Exercise 1 Let φ be a formula of type σ in the language of HA^ω. Show that HA^ω proves that

$$\mathbf{R}^\sigma\,\mathbf{mr}\,\big[\,\varphi(0) \to \big(\,\forall x^0\,(\varphi(x) \to \varphi(Sx)\,) \to \forall x^0 \varphi(x)\,\big)\,\big].$$

Exercise 2 Let $\varphi(x)$ be a formula of type σ and ψ be a formula of type τ in the language of HA^ω (the variable x is of type ρ and does occur freely in $\varphi(x)$, but not in ψ) . Show that

$$t\,\mathbf{mr}\,\big[\,\big(\,\exists x^\rho\,\varphi(x) \to \psi\,\big) \to \forall x^\rho\,\big(\,\varphi(x) \to \psi\,\big)\big]$$

where $t = \lambda s^{(\rho\times\sigma)\to\tau}.\lambda x^\rho.\lambda y^\sigma.s(\mathbf{p}xy)$.

Exercise 3 In this exercise we work in HA^ω.

(a) Let φ be a formula of type τ whose free variables are x^ρ and y^σ. Show that HA^ω proves that

$$t\,\mathbf{mr}\,\big[\,\exists x^\rho\,\forall y^\sigma\,\varphi(x,y) \to \forall y^\sigma\,\exists x^\rho\,\varphi(x,y)\,\big]$$

if $t = \lambda s^{\rho\times(\sigma\to\tau)}.\lambda a^\sigma.\mathbf{p}(\mathbf{p}_0 s)(\mathbf{p}_1 sa)$.

(b) Let φ, ψ and χ be sentences. Construct a term t such that HA^ω proves that

$$t\,\mathbf{mr}\,\big[\,(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)\,\big].$$

Do not just give right term but also show that it is correct!

Exercise 4 (For people familiar with recursion theory.) Recall that we can code pairs of natural numbers as natural numbers in such a way that both the pairing and the projection operations are computable, and let us write $\langle m, n \rangle$ for a natural number coding the pair consisting of natural numbers m and n. In addition, we assume that we have fixed some suitable enumeration of the partial computable functions from the natural numbers to the natural numbers. We will write $m \cdot n \downarrow$ to mean: the mth computable function terminates on input n, in which case we will write $m \cdot n$ for the result.

By induction on the finite type σ we will define a set of natural numbers HEO_σ and an equivalence relation $\sim_\sigma$ on that set.

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\begin{array}{rcl}
\mathrm{HEO}_0 & = & \mathbb{N} \\
m \sim_0 n & \Leftrightarrow & m = n \\
\mathrm{HEO}_{\sigma \times \tau} & = & \{\langle m, n \rangle : m \in \mathrm{HEO}_\sigma, n \in \mathrm{HEO}_\tau\} \\
\langle m, n \rangle \sim_{\sigma \times \tau} \langle m', n' \rangle & \Leftrightarrow & m \sim_\sigma m' \text{ and } n \sim_\tau n' \\
\mathrm{HEO}_{\sigma \to \tau} & = & \{n \in \mathbb{N} \colon \forall m \in \mathrm{HEO}_\sigma \, (n \cdot m \downarrow \wedge n \cdot m \in \mathrm{HEO}_\tau) \text{ and} \\
& & \forall m, m' \in \mathrm{HEO}_\sigma \, (\, m \sim_\sigma m' \to n \cdot m \sim_\tau n \cdot m')\} \\
n \sim_{\sigma \to \tau} n' & \Leftrightarrow & (\forall m \in \mathrm{HEO}_\sigma) \, n \cdot m \sim_\tau n \cdot m'
\end{array}
$$

(a) Convince yourself that $\sim_\sigma$ does indeed define an equivalence relation on HEO_σ.

(b) Show that HEO can be regarded as a model of $\mathsf{E\text{-}PA}^\omega$ if we interpret elements of type σ as elements of HEO_σ and equality of objects of type σ as $\sim_\sigma$.

(c) Show that in the HEO-model of $\mathsf{E\text{-}PA}^\omega$ the following choice principle fails:

$$\mathsf{AC}_{0,0}\colon \quad \forall x^0 \, \exists y^0 \, \varphi(x, y) \to \exists f^{0 \to 0} \, \forall x^0 \, \varphi(x, f(x)).$$