10th Exercise sheet Proof Theory
14 Mar 2018

Exercise 1 The handout constructs a closed term **plus** of type $0 \to (0 \to 0)$ such that HA^ω proves

$$\begin{aligned} \mathbf{plus}\, 0\, n &= n \\ \mathbf{plus}\, (Sm)\, n &= S(\mathbf{plus}\, m\, n) \end{aligned}$$

(a) Show $\mathsf{HA}^\omega \vdash \forall x^0, y^0 \left(\mathbf{plus}\, x\, (Sy) = S(\mathbf{plus}\, x\, y) \right)$.

Hint: Write $\varphi :\equiv \forall y^0 \left(\mathbf{plus}\, x\, (Sy) = S(\mathbf{plus}\, x\, y) \right)$ and prove $\forall x^0\, \varphi$ by induction on x. And just use the equations above (and not the definition of **plus**).

(b) Show that $\mathsf{HA}^\omega \vdash \forall x^0, y^0 \,(\, \mathbf{plus}\, x\, y = \mathbf{plus}\, y\, x \,)$.

Exercise 2 (a) Construct a closed term **times** of type $0 \to (0 \to 0)$ such that HA^ω proves

$$\begin{aligned} \mathbf{times}\, 0\, n &= 0 \\ \mathbf{times}\, (Sm)\, n &= \mathbf{plus}\, (\mathbf{times}\, m\, n)\, n \end{aligned}$$

(b) Show $\mathsf{HA}^\omega \vdash \forall x^0, y^0 \,(\, \mathbf{times}\, x\, y = \mathbf{times}\, y\, x \,)$.

Exercise 3 (a) Construct a closed term **fact** of type $0 \to 0$ such that HA^ω proves

$$\begin{aligned} \mathbf{fact}\, 0 &= S0 \\ \mathbf{fact}\, (Sn) &= \mathbf{times}\, (\mathbf{fact}\, n)\, (Sn) \end{aligned}$$

(b) Construct a closed term **pred** of type $0 \to 0$ such that HA^ω proves

$$\begin{aligned} \mathbf{pred}\, 0 &= 0 \\ \mathbf{pred}\, (Sn) &= n. \end{aligned}$$

Exercise 4 Let ∇ be a function mapping formulas in the language of HA^ω to formulas in the language in HA^ω, such that the following statements are provable in HA^ω:

$$\begin{array}{l} \mathsf{HA}^\omega \vdash \varphi \to \nabla\varphi \\ \mathsf{HA}^\omega \vdash \nabla(\varphi \wedge \psi) \leftrightarrow (\nabla\varphi \wedge \nabla\psi) \\ \mathsf{HA}^\omega \vdash (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi) \\ \mathsf{HA}^\omega \vdash (\nabla\varphi)[t/x] \leftrightarrow \nabla(\varphi[t/x]) \end{array}$$

In addition, let φ^∇ be the formula obtained from φ by applying ∇ to each atomic subformula, disjunction and existentially quantified subformula. More precisely, φ^∇ is defined by induction on the structure of φ as follows:

$$\begin{array}{rcll} \varphi^\nabla & := & \nabla\varphi & \text{if } \varphi \text{ is an atomic formula,} \\ (\varphi \wedge \psi)^\nabla & := & \varphi^\nabla \wedge \psi^\nabla, & \\ (\varphi \vee \psi)^\nabla & := & \nabla(\varphi^\nabla \vee \psi^\nabla), & \\ (\varphi \to \psi)^\nabla & := & \varphi^\nabla \to \psi^\nabla, & \\ (\forall x^\sigma\, \varphi)^\nabla & := & \forall x^\sigma\, \varphi^\nabla, & \\ (\exists x^\sigma\, \varphi)^\nabla & := & \nabla\exists x^\sigma\, \varphi^\nabla. & \end{array}$$

Show that $\mathsf{HA}^\omega \vdash \varphi$ implies $\mathsf{HA}^\omega \vdash \varphi^\nabla$.

Exercise 5 (Tricky!) Construct a closed term $\mathbf{A}$ of type $0 \to (0 \to 0)$ that satisfies the defining equations of the Ackermann function. That is, construct a term $\mathbf{A}$ such that HA^ω proves that

$$\begin{array}{rcl} \mathbf{A}\, 0\, y & = & Sy, \\ \mathbf{A}\, (Sx)\, 0 & = & \mathbf{A}\, x\, (S0) \\ \mathbf{A}\, (Sx)\, (Sy) & = & \mathbf{A}\, x\, (\, A\, (Sx)\, y\,). \end{array}$$