2nd Homework sheet Proof Theory

- Deadline: 17 November, 9:00 sharp.
- Submit your solutions by handing them to the lecturer or the teaching assistant at the *beginning of the lecture*.
- The homework exercise continues on the next page.
- Good luck!

Exercise 1 In this exercise we will be working in propositional logic, so we will only consider propositional formulas.

A formula φ is in *negation normal form* if all implications which occur in φ have a propositional variable or $\bot$ on the left and $\bot$ on the right. In classical logic every formula is equivalent to a formula in negation normal form. One way of seeing this is as follows: define for any propositional formula φ two new formulas $\mathbf{T}\varphi$ and $\mathbf{F}\varphi$ by simultaneous recursion, as follows:

$$\begin{array}{lcll}
\mathbf{T}\varphi & = & \varphi & \text{if } \varphi \text{ is a propositional variable or } \bot \\
\mathbf{T}(\varphi \wedge \psi) & = & \mathbf{T}\varphi \wedge \mathbf{T}\psi & \\
\mathbf{T}(\varphi \vee \psi) & = & \mathbf{T}\varphi \vee \mathbf{T}\psi & \\
\mathbf{T}(\varphi \to \psi) & = & \mathbf{F}\varphi \vee \mathbf{T}\psi & \\
 & & & \\
\mathbf{F}\varphi & = & \neg\varphi & \text{if } \varphi \text{ is a propositional variable or } \bot \\
\mathbf{F}(\varphi \wedge \psi) & = & \mathbf{F}\varphi \vee \mathbf{F}\psi & \\
\mathbf{F}(\varphi \vee \psi) & = & \mathbf{F}\varphi \wedge \mathbf{F}\psi & \\
\mathbf{F}(\varphi \to \psi) & = & \mathbf{T}\varphi \wedge \mathbf{F}\psi &
\end{array}$$

It is easy to see that for any formula φ both $\mathbf{T}\varphi$ and $\mathbf{F}\varphi$ are in negation normal form and that $\mathbf{T}\varphi$ is classically equivalent to φ, while $\mathbf{F}\varphi$ is classically equivalent to $\neg\varphi$ (you do not need to prove these facts).

(a) *(20 points)* Show that for every formula φ there is a derivation of $\mathbf{T}\varphi, \mathbf{F}\varphi \vdash \bot$ in intuitionistic natural deduction.

(b) *(30 points)* Prove the following implication: if the sequent $\varphi_1, \dots, \varphi_n \Rightarrow \psi_1, \dots, \psi_m$ is provable in the classical sequent calculus without the cut rule, then there is a derivation of $\mathbf{T}\varphi_1, \dots, \mathbf{T}\varphi_n, \mathbf{F}\psi_1, \dots, \mathbf{F}\psi_m \vdash \bot$ in intuitionistic natural deduction.

(c) *(20 points)* Define $\varphi^* = \neg\mathbf{F}\varphi$. Deduce from (b) and the completeness of the classical sequent calculus without the cut rule that φ is a classical tautology precisely when φ^* is an intuitionistic tautology.

(d) *(30 points)* Is the mapping $\varphi \mapsto \varphi^*$ defined in (c) a negative translation? Justify your answer!