7th Exercise sheet Proof Theory
13 Dec 2016

Exercise 1 Let φ be a formula of type σ in the language of HA^ω. Show that HA^ω proves that

$$\mathbf{R}^\sigma \, \mathbf{mr} \left[\varphi(0) \to \left(\forall x^0 \left(\varphi(x) \to \varphi(Sx) \right) \to \forall x^0 \varphi(x) \right) \right].$$

Exercise 2 Let $\varphi(x)$ be a formula of type σ and ψ be a formula of type τ in the language of HA^ω (the variable x is of type ρ and does occur freely in $\varphi(x)$, but not in ψ) . Show that

$$t \, \mathbf{mr} \left[\left(\exists x^\rho \, \varphi(x) \to \psi \right) \to \forall x^\rho \left(\varphi(x) \to \psi \right) \right]$$

where $t = \lambda s^{(\rho \times \sigma) \to \tau} . \lambda x^\rho . \lambda y^\sigma . s(\mathbf{p} x y)$.

Exercise 3 In this exercise we work in HA^ω.

(a) Let φ be a formula of type τ whose free variables are x^ρ and y^σ. Show that HA^ω proves that

$$t \, \mathbf{mr} \left[\exists x^\rho \, \forall y^\sigma \, \varphi(x, y) \to \forall y^\sigma \, \exists x^\rho \, \varphi(x, y) \right]$$

if $t = \lambda s^{\rho \times (\sigma \to \tau)} . \lambda a^\sigma . \mathbf{p}(\mathbf{p}_0 s)(\mathbf{p}_1 s a)$.

(b) Let φ, ψ and χ be sentences. Construct a term t such that HA^ω proves that

$$t \, \mathbf{mr} \left[(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi) \right].$$

Do not just give right term but also show that it is correct!

Exercise 4 (a) Show that for any simple formula φ with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ there is a closed term $\mathbf{d}$ of type $\sigma \to (\sigma_2 \to \ldots \to (\sigma_n \to 0))$ in the language of HA^ω such that

$$\mathsf{HA}^\omega \vdash \forall x_1^{\sigma_1}, \ldots, x_n^{\sigma_n} \left(\varphi \leftrightarrow \mathbf{d} x_1 \ldots x_n =_0 0 \right).$$

Hint: Use induction on the structure of φ.

(b) Deduce that for any simple formula φ with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ we have

$$\mathsf{HA}^\omega \vdash \forall x_1^{\sigma_1}, \ldots, x_n^{\sigma_n} \left(\varphi \vee \neg\varphi \right).$$