6th Exercise sheet Proof Theory
6 Dec 2016

Exercise 1 In the lecture we showed that there is a closed term **plus** of type $0 \to (0 \to 0)$ such that HA^ω proves

$$\begin{aligned} \mathbf{plus}\, m\, 0 &= m \\ \mathbf{plus}\, m\, Sn &= S(\mathbf{plus}\, m\, n) \end{aligned}$$

(a) Show $\mathsf{HA}^\omega \vdash \forall x^0, y^0 \left(S(\mathbf{plus}\, y\, x) = \mathbf{plus}\, Sy\, x \right)$.

Hint: Write $\varphi := \forall y^0 \left(S(\mathbf{plus}\, y\, x) = \mathbf{plus}\, Sy\, x \right)$ and prove $\forall x^0\, \varphi$ by induction on x. And just use the equations above (and not the definition of **plus**).

(b) Show that $\mathsf{HA}^\omega \vdash \forall x^0, y^0\, (\,\mathbf{plus}\, x\, y = \mathbf{plus}\, y\, x\,)$.

Exercise 2 (a) Construct a closed term **times** of type $0 \to (0 \to 0)$ such that HA^ω proves

$$\begin{aligned} \mathbf{times}\, m\, 0 &= 0 \\ \mathbf{times}\, m\, Sn &= \mathbf{plus}\, (\mathbf{times}\, m\, n)\, m \end{aligned}$$

(b) Show $\mathsf{HA}^\omega \vdash \forall x^0, y^0\, (\,\mathbf{times}\, x\, y = \mathbf{times}\, y\, x\,)$.

Exercise 3 (a) Construct a closed term **fact** of type $0 \to 0$ such that HA^ω proves

$$\begin{aligned} \mathbf{fact}\, 0 &= S0 \\ \mathbf{fact}\, Sn &= \mathbf{times}\, (\mathbf{fact}\, n)\, Sn \end{aligned}$$

(b) Construct a closed term **pred** of type $0 \to 0$ such that HA^ω proves

$$\begin{aligned} \mathbf{pred}\, 0 &= 0 \\ \mathbf{pred}\, Sn &= n. \end{aligned}$$

Exercise 4 Show that in Gödel's $\mathcal{T}$ the only closed terms of type 0 in normal form are numerals, that is, expressions of the form $S^m 0$ for some natural number m.

Exercise 5 Let ∇ be a function mapping formulas in the language of HA^ω to formulas in the language in HA^ω, such that the following statements are provable in HA^ω:

$$\begin{array}{l} \mathsf{HA}^\omega \vdash \varphi \to \nabla\varphi \\ \mathsf{HA}^\omega \vdash \nabla(\varphi \wedge \psi) \leftrightarrow (\,\nabla\varphi \wedge \nabla\psi\,) \\ \mathsf{HA}^\omega \vdash (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi) \end{array}$$

In addition, let φ^∇ be the formula obtained from φ by applying ∇ to each atomic subformula, disjunction and existentially quantified subformula. More precisely, φ^∇ is defined by induction on the structure of φ as follows:

$$\begin{array}{rcll} \varphi^\nabla & := & \nabla\varphi & \text{if } \varphi \text{ is an atomic formula,} \\ (\varphi \wedge \psi)^\nabla & := & \varphi^\nabla \wedge \psi^\nabla, & \\ (\varphi \vee \psi)^\nabla & := & \nabla(\varphi^\nabla \vee \psi^\nabla), & \\ (\varphi \to \psi)^\nabla & := & \varphi^\nabla \to \psi^\nabla, & \\ (\forall x^\sigma\, \varphi(x)\,)^\nabla & := & \forall x^\sigma\, (\varphi(x))^\nabla, & \\ (\exists x^\sigma\, \varphi(x)\,)^\nabla & := & \nabla\exists x^\sigma.\, (\varphi(x))^\nabla. & \end{array}$$

Show that $\mathsf{HA}^\omega \vdash \varphi$ implies $\mathsf{HA}^\omega \vdash \varphi^\nabla$.

Exercise 6 (Tricky!) Construct a closed term $\mathbf{A}$ of type $0 \to (0 \to 0)$ that satisfies the defining equations of the Ackermann function, i.e., a term $\mathbf{A}$ such that HA^ω proves that

$$\begin{array}{rcl} \mathbf{A}(0, y) & = & Sy, \\ \mathbf{A}(Sx, 0) & = & \mathbf{A}(x, 1), \\ \mathbf{A}(Sx, Sy) & = & \mathbf{A}(x, A(Sx, y)). \end{array}$$