1st Exercise sheet Proof Theory
3 Nov 2016

Exercise 1 Consider the following De Morgan laws:

$$\neg(\varphi \vee \psi) \to \neg\varphi \wedge \neg\psi$$
$$\neg\varphi \wedge \neg\psi \to \neg(\varphi \vee \psi)$$
$$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$$
$$\neg\varphi \vee \neg\psi \to \neg(\varphi \wedge \psi)$$

Which ones of these are also intuitionistic tautologies? Justify your answer using Kripke models.

Exercise 2 Peirce's Law is the formula

$$((p \to q) \to p) \to p.$$

Show that this principle is a classical, but not an intuitionistic tautology.

Exercise 3 (a) Let (W, R, f) be a Kripke model and let $\sim$ be the relation on W defined by: $x \sim y$ if $R(x, y)$ and $R(y, x)$. Check that $\sim$ is an equivalence relation and write $[w]$ for the $\sim$-equivalence class of w. Show that that there is a Kripke model $(W/\sim, R', f')$ with set of worlds $W/\sim$, such that $[w] \Vdash \varphi$ in $(W/\sim, R', f')$ if and only if $w \Vdash \varphi$ in (W, R, f).

(b) Using part (a) and assuming the completeness of Kripke semantics with respect to intuitionistic propositional logic, show that the Kripke models (W, R, f) in which R is not only reflexive and transitive, but also anti-symmetric, also form a complete semantics for intuitionistic propositional logic.

Exercise 4 Give a semantic proof of the fact that intuitionistic propositional logic has the disjunction property: if $\varphi \vee \psi$ is an intuitionistic tautology, then so is at least one of φ and ψ. Why does this fail for classical logic?