CHAPTER 10

Semantics of classical predicate logic

1. Syntax of predicate logic

The symbols of predicate logic include:

- A countable set of *parameters* Par.
- A countable set of *variables* Var.
- A countable set of relation symbols $\mathcal{R}$, where each relation symbol has an *arity* which is a certain natural number. Relation symbols of arity 0 are called *propositional variables*.
- A set of function symbols $\mathcal{F}$, where each function symbol has an *arity* which is a certain natural number. Function symbols of arity 0 are called *constants*.
- A special propositional constant $\bot$.
- The logical connectives $\wedge, \vee, \to$.
- In addition, there are also quantifiers $\forall$ and $\exists$.
- We include the brackets (and) as well as the comma.

Definition 1.1. The collection of *semi-terms* is defined inductively as follows:

- each parameters or variable is a semi-term.
- if $t_1, \ldots, t_n$ are semi-terms and f is a function symbol, then $f(t_1, \ldots, t_n)$ is a semi-term.

The collection of *semi-formulas* is defined inductively as follows:

- $\bot$ is a semi-formula.
- if $t_1, \ldots, t_n$ are semi-terms and R is an n-ary relation symbol, then $R(t_1, \ldots, t_n)$ is a semi-formula.
- if φ and ψ are semi-formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi$ and $\varphi \to \psi$.
- if φ is a semi-formula, then so are $\forall x\, \varphi$ and $\exists x\, \varphi$ for any *variable* x.

Definition 1.2. The collection $F(t)$ for a semi-term t is defined inductively as:

- $F(t) = \{t\}$ if t is a parameter or variable.
- $F(f(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} F(t_i)$.

The collection $F(\varphi)$ for a semi-formula φ is defined inductively as:

- $F(R(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} F(t_i)$.
- $F(\varphi \Box \psi) = F(\varphi) \cup F(\psi)$ for $\Box \in \{\wedge, \vee, \to\}$.
- $F(\exists x\, \varphi) = F(\forall x\, \varphi) = F(\varphi) \setminus \{x\}$.

DEFINITION 1.3. A *term* is a semi-term t such that the set $F(t)$ contains only parameters. These parameters are said to *occur* in t. Similarly, a *formula* is a semi-formula φ such that the set $F(\varphi)$ contains only parameters. These parameters are said to *occur* in φ.

If φ is a semi-formula with $x \in F(\varphi)$ and t is a term, then $\varphi[t/x]$ is the result of substituting t for x (that is, replacing every occurrence of x with t). If x is clear from the context, we will also just write $\varphi(t)$.

2. Models for classical predicate logic

DEFINITION 2.1. A (classical) model $\mathcal{M}$ consists of:

- a non-empty set M.
- for each n-ary relation symbol R a relation $R^M \subseteq M^n$.
- for each n-ary function symbol f a function $f^M : M^n \to M$.

If $\mathcal{M}$ and $\mathcal{N}$ are models, then a *homomorphism* from $\mathcal{M}$ to $\mathcal{N}$ is a function $\tau : M \to N$ such that:

- for every n-ary relation symbol R and $m_1, \ldots, m_n \in M$, we have
$$(m_1, \ldots, m_n) \in R^M \Longrightarrow (\tau(m_1), \ldots, \tau(m_n)) \in R^N.$$
- for every n-ary function symbol f and $m_1, \ldots, m_n \in M$, we have
$$\tau(f^M(m_1, \ldots, m_n)) = f^N(\tau(m_1), \ldots, \tau(m_n)).$$

DEFINITION 2.2. If $\mathcal{M}$ is a model, then an *assignment* for $\mathcal{M}$ is a function $\alpha : \mathrm{Par} \to M$. If α is an assignment for $\mathcal{M}$, a is a parameter and $m \in M$, then $\alpha[m/a]$ is the assigment defined by:

$$\alpha[m/a](b) \quad = \quad \begin{cases} \alpha(b) & \text{if } b \neq a \\ m & \text{if } b = a \end{cases}$$

DEFINITION 2.3. If t is a term and $\mathcal{M}$ is a model together with an assignment α, then the interpretation in $\mathcal{M}$ of the term t under the assignment α, written $I_\alpha^{\mathcal{M}}(t)$, is defined inductively as:

- $I_\alpha^{\mathcal{M}}(a) = \alpha(a)$ if a is a parameter.
- $I_\alpha^{\mathcal{M}}(f(t_1, \ldots, t_n)) = f^M(I_\alpha^{\mathcal{M}}(t_1), \ldots, I_\alpha^{\mathcal{M}}(t_n))$.

DEFINITION 2.4. If φ is a formula, $\mathcal{M}$ is a model and α is assignment for $\mathcal{M}$, then $\mathcal{M} \models \varphi[\alpha]$, to be pronounced: φ is true in $\mathcal{M}$ under the assignment α, is defined inductively as follows:

$$\begin{array}{rcl}
\mathcal{M} \models \bot[\alpha] & \Leftrightarrow & \text{Never!} \\
\mathcal{M} \models R(t_1, \ldots, t_n)[\alpha] & \Leftrightarrow & (I_\alpha^{\mathcal{M}}(t_1), \ldots, I_\alpha^{\mathcal{M}}(t_n)) \in R^M \\
\mathcal{M} \models (\varphi \wedge \psi)[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha] \text{ and } \mathcal{M} \models \psi[\alpha] \\
\mathcal{M} \models (\varphi \vee \psi)[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha] \text{ or } \mathcal{M} \models \psi[\alpha] \\
\mathcal{M} \models (\varphi \to \psi)[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi[\alpha] \text{ implies } \mathcal{M} \models \psi[\alpha] \\
\mathcal{M} \models \exists x\, \varphi[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi(a)[\alpha[m/a]] \text{ for some parameter } a \\
 & & \text{ not occurring in } \varphi \text{ and } m \in M \\
\mathcal{M} \models \forall x\, \varphi[\alpha] & \Leftrightarrow & \mathcal{M} \models \varphi(a)[\alpha[m/a]] \text{ for all parameters } a \\
 & & \text{ not occurring in } \varphi \text{ and } m \in M
\end{array}$$

Note that the truth of $\mathcal{M} \models \varphi[\alpha]$ depends only on what α does on parameters occurring in φ: so if β is another assignment and $\alpha \restriction F(\varphi) = \beta \restriction F(\varphi)$, then

$$\mathcal{M} \models \varphi[\alpha] \text{ if and only if } \mathcal{M} \models \varphi[\beta].$$

Definition 2.5. We will write $\mathcal{M} \models \varphi$ and say that φ holds in $\mathcal{M}$, if $\mathcal{M} \models \varphi[\alpha]$ for any assignment α. Moreover, we will say write $\models \varphi$ and say that φ is a *classical tautology* if φ holds in all models. We will write $\Gamma \models \Delta$ if for any model $\mathcal{M}$ and any assignment α such that all formulas in Γ are true in $\mathcal{M}$ under the assignment α, at least one formula in Δ is also true in $\mathcal{M}$ under the assignment α; the special case $\Gamma \models \{\varphi\}$ is usually just written $\Gamma \models \varphi$.

3. Consistency properties for classical predicate logic

As expected, we extend the notion of signed formulas to predicate logic. Their validity in a model $\mathcal{M}$ under an assigment α is extended in the obvious way:

$$\begin{aligned} \mathcal{M} \models \mathbf{t}\varphi[\alpha] \quad &\Leftrightarrow \quad \mathcal{M} \models \varphi[\alpha] \\ \mathcal{M} \models \mathbf{f}\varphi[\alpha] \quad &\Leftrightarrow \quad \mathcal{M} \not\models \varphi[\alpha] \end{aligned}$$

Literals are now defined as formulas of the form $\mathbf{t}\varphi$ and $\mathbf{f}\varphi$, where φ is an atomic formula (that is, $\perp$ or a formula of the form $R(t_1, \ldots, t_n)$). Also, we now have two new classes of formulas:

Definition 3.1. The *γ-formulas* are those of the form $\mathbf{t}\,\forall x\,\varphi(x)$ and $\mathbf{f}\,\exists x\,\varphi(x)$. If γ is a formula of one of these two forms and t is a term, then $\gamma(t)$ is the signed formula given by the following table:

γ	$\gamma(t)$
$\mathbf{t}\,\forall x\,\varphi$	$\mathbf{t}\,\varphi[t/x]$
$\mathbf{f}\,\exists x\,\varphi$	$\mathbf{f}\,\varphi[t/x]$

The *δ-formulas* are those of the form $\mathbf{t}\,\exists x\,\varphi(x)$ and $\mathbf{f}\,\forall x\,\varphi(x)$. If δ is a formula of one of these two forms and t is a term, then $\delta(t)$ is the signed formula given by the following table:

δ	$\delta(t)$
$\mathbf{t}\,\exists x\,\varphi$	$\mathbf{t}\,\varphi[t/x]$
$\mathbf{f}\,\forall x\,\varphi$	$\mathbf{f}\,\varphi[t/x]$

Definition 3.2. Let $\mathcal{C}$ be a collection of sets of signed formulas. $\mathcal{C}$ will be called *consistency property (for classical predicate logic)*, if for any $\Gamma \in \mathcal{C}$, we have:

(1) Γ does not contain both a literal and its dual.
(2) $\mathbf{t}\perp \notin \Gamma$.
(3) if $\sigma \in \Gamma$ and σ is an α-formula, then also $\Gamma, \sigma_1, \sigma_2 \in \mathcal{C}$.
(4) if $\sigma \in \Gamma$ and σ is a β-formula, then $\Gamma, \sigma_1 \in \mathcal{C}$ or $\Gamma, \sigma_1 \in \mathcal{C}$.
(5) if $\sigma \in \Gamma$ and σ is a γ-formula, then also $\Gamma, \sigma(t) \in \mathcal{C}$ for any term t.
(6) if $\sigma \in \Gamma$ and σ is a δ-formula, then also $\Gamma, \sigma(a) \in \mathcal{C}$ for any parameter not occurring in Γ.

Again we can make the following observations:

(1) if $\mathcal{C}$ is a consistency property and $\Gamma \in \mathcal{C}$, then Γ cannot contain both a signed formula and its dual.
(2) any consistency property can be extended to one of *finite character*, where a consistency property $\mathcal{C}$ is of finite character if for any collection Γ of signed formulas we have: $\Gamma \in \mathcal{C}$ if and only if $\Gamma_0 \in \mathcal{C}$ for every finite subset $\Gamma_0 \subseteq \Gamma$.

THEOREM 3.3. (Fundamental theorem on consistency properties for classical predicate logic) *If $\mathcal{C}$ is a consistency property for classical predicate logic, $\Gamma \in \mathcal{C}$ and there are infinitely many parameters not occurring in Γ, then there are a classical model $\mathcal{M}$ and an assignment α for $\mathcal{M}$ such that all formulas in Γ are valid in $\mathcal{M}$ under the assignment α.*

We follow the usual pattern, making the necessary changes.

DEFINITION 3.4. We will call a non-empty set of parameters $U \subseteq \text{Par}$ such that there are infinitely many parameters that do not belong to U a *universe*. A *Hintikka set (for classical predicate logic)* relative to a universe U is a collection of signed formulas Γ such that the following hold:

(1) all the parameters in Γ come from U.
(2) Γ does contain both a literal and its dual.
(3) $\mathbf{t}\bot \notin \Gamma$.
(4) if $\sigma \in \Gamma$ and σ is an α-formula, then also $\sigma_1, \sigma_2 \in \Gamma$.
(5) if $\sigma \in \Gamma$ and σ is a β-formula, then $\sigma_1 \in \Gamma$ or $\sigma_2 \in \Gamma$.
(6) if $\sigma \in \Gamma$ and σ is a γ-formula, then also $\sigma(t) \in \Gamma$ for any term t in which only parameters from U occur.
(7) if $\sigma \in \Gamma$ and σ is a δ-formula, then also $\sigma(a) \in \Gamma$ for a parameter $a \in U$.

LEMMA 3.5. *If $\mathcal{C}$ is a consistency property of finite character, $\Gamma \in \mathcal{C}$ and all parameters from Γ belong to a universe U, then there is a universe U_∞ and a Hintikka set Γ_∞ relative to U_∞ with $U \subseteq U_\infty$, $\Gamma \subseteq \Gamma_\infty$ and $\Gamma_\infty \in \mathcal{C}$.*

PROOF. We first choose a suitable U_∞: since the complement of U in Par is infinite, it can be partitioned in two infinite sets U_1 and U_2. Set $U_\infty := U \cup U_1$ and let $c_0, c_1, c_2, \ldots$ be an enumeration of U_1.

The idea of the proof is again to create an increasing sequence of sets of signed formulas Γ_n with the following properties:

(1) $\Gamma_0 = \Gamma$.
(2) Each Γ_n belongs to $\mathcal{C}$.
(3) Each Γ_n contains only finitely many parameters from U_1.
(4) If $\Gamma_\infty = \bigcup \Gamma_n$, then Γ_∞ is a Hintikka set relative to U_∞.

Since property 2 and the finite character of $\mathcal{C}$ imply that $\Gamma_\infty \in \mathcal{C}$, this would prove the result.

Obviously, we start by putting $\Gamma_0 = \Gamma$. Since both

- the collection of signed formulas σ with parameters from U_∞, which are not literals, and
- the collection of terms with parameters from U_∞

are countable, we can arrange for enumerations $\sigma_0, \sigma_1, \ldots$ of such formulas and $t_0, t_1, t_2, \ldots$ of such terms, in such a way that in the combined sequence $(\sigma_n, t_n)_n$ we see each combination of a non-literal and term with parameters from U_∞ infinitely often.

We define Γ_{n+1} once Γ_n has been defined, as follows:

(1) if $\sigma_n \notin \Gamma_n$, then $\Gamma_{n+1} = \Gamma_n$.
(2) if $\sigma_n \in \Gamma_n$ and σ_n is of α-type, then $\Gamma_{n+1} = \Gamma_n, (\sigma_n)_1, (\sigma_n)_2$.
(3) if $\sigma_n \in \Gamma_n$ and σ_n is of β-type and $\Gamma_n, (\sigma_n)_1 \in \mathcal{C}$, then $\Gamma_{n+1} = \Gamma_n, (\sigma_n)_1$.
(4) if $\sigma_n \in \Gamma_n$ and σ_n is of β-type and $\Gamma_n, (\sigma_n)_1 \notin \mathcal{C}$, then $\Gamma_{n+1} = \Gamma_n, (\sigma_n)_2$.
(5) if $\sigma_n \in \Gamma_n$ and σ_n is of γ-type, then $\Gamma_{n+1} = \Gamma_n, \sigma_n(t_n)$.
(6) if $\sigma_n \in \Gamma_n$ and σ_n is of δ-type, then $\Gamma_{n+1} = \Gamma_n, \sigma_n(a)$ where a is the c_i with smallest index which does not occur in Γ_n.

One readily checks that this sequence has the required properties. □

PROOF. (Of Theorem 3.3.) Without loss of generality we may assume that $\mathcal{C}$ is a consistency property of finite character. The parameters from Γ belong to some universe U, so by the previous lemma there is a universe U_∞ extending U and a Hintikka set Γ_∞ relative to U_∞ extending Γ with $\Gamma_\infty \in \mathcal{C}$.

Now construct a model $\mathcal{M}$ as follows: the elements of the model are the terms with parameters from U_∞, and $(t_1, \ldots, t_n)$ will belongs to the interpretation of the n-ary relation symbol R precisely when $\mathbf{t}P(t_1, \ldots, t_n)$ belongs to Γ_∞, while the interpretation of the n-ary function symbol f is the function sending the n-tuple of terms $(t_1, \ldots, t_n)$ to $f(t_1, \ldots, t_n)$. The assignment α we need is obtained by sending elements in U_∞ to themselves, while doing something completely arbitrary outside U_∞.

Now one readily proves by induction on the structure of the terms that they are interpreted by themselves, while a proof by induction of the structure of the signed formula σ can be used to show that $\mathcal{M} \models \sigma[\alpha]$ whenever $\sigma \in \Gamma_\infty$. So all sentences in Γ are true in $\mathcal{M}$ under the assignment α. □

CHAPTER 11

Semantics of intuitionistic predicate logic

1. Kripke models for intuitionistic predicate logic

DEFINITION 1.1. A *Kripke model for intuitionistic predicate logic* is a quadruple (W, R, f, τ) such that:

- W is a non-empty set ("the set of worlds").
- R is a reflexive and transitive relation.
- f is a function assigning to every world $w \in W$ a classical model $f(w)$; instead of $f(w)$, we will frequently write $\mathcal{M}_w$ when it is clear from the context which Kripke model we mean.
- τ assigns to every pair $(w, w') \in R$ a homomorphism of models $\tau_{ww'}: \mathcal{M}_w \to \mathcal{M}_{w'}$, such that $\tau_{ww} = \mathsf{id}_{\mathcal{M}_w}$ for every $w \in W$ and $\tau_{w',w''} \circ \tau_{w,w'} = \tau_{w,w''}$, whenever wRw' and $w'Rw''$.

Note that if (W, R, f, τ) is a Kripke model, $w \in W$ and α is an assignment for $\mathcal{M}_w$, then α determines an assignment for every w' with wRw', simply by postcomposition with $\tau_{ww'}$; we will denote this assignment by $\alpha_{w'}$ (so $\alpha_{w'} := \tau_{ww'} \circ \alpha$).

DEFINITION 1.2. If (W, R, f, τ) is a Kripke model, $w \in W$ a world, α an assignment for $\mathcal{M}_w$ and φ is a first-order formula, then we define $w \Vdash \varphi[\alpha]$ by induction on φ as follows:

$$
\begin{aligned}
w \Vdash \varphi[\alpha] \quad &:\Leftrightarrow \quad \mathcal{M}_w \models \varphi[\alpha], \text{ whenever } \varphi \text{ is atomic} \\
w \Vdash (\varphi \wedge \psi)[\alpha] \quad &:\Leftrightarrow \quad w \Vdash \varphi[\alpha] \text{ and } w \Vdash \psi[\alpha] \\
w \Vdash (\varphi \vee \psi)[\alpha] \quad &:\Leftrightarrow \quad w \Vdash \varphi[\alpha] \text{ or } w \Vdash \psi[\alpha] \\
w \Vdash (\varphi \to \psi)[\alpha] \quad &:\Leftrightarrow \quad (\forall w' \in W) \text{ if } wRw' \text{ and } w' \Vdash \varphi[\alpha_{w'}], \text{ then } w' \Vdash \psi[\alpha_{w'}] \\
w \Vdash (\exists x\, \varphi)[\alpha] \quad &:\Leftrightarrow \quad \text{there is a parameter } a \text{ not occurring in } \varphi \text{ and } m \in \mathcal{M}_w \\
&\qquad\quad \text{such that } w \Vdash \varphi(a)[\alpha[a/m]] \\
w \Vdash (\forall x\, \varphi)[\alpha] \quad &:\Leftrightarrow \quad (\forall w' \in W) \text{ if } wRw', a \text{ does not occur in } \varphi \\
&\qquad\quad \text{and } m \in \mathcal{M}_{w'}, \text{ then } w' \Vdash \varphi(a)[\alpha_{w'}[m/a]]
\end{aligned}
$$

LEMMA 1.3. (Monotonicity) *If (W, R, f, τ) is a Kripke model, $w, w' \in W$ are two worlds such that wRw' and α is an assignment for $\mathcal{M}_w$, then $w \Vdash \varphi[\alpha]$ implies $w' \Vdash \varphi[\alpha_{w'}]$.*

DEFINITION 1.4. Let (W, R, f, τ) be a Kripke model and $w \in W$. If $w \Vdash \varphi[\alpha]$ for all assignment α, then we will write $w \Vdash \varphi$. We will write $\models_{\mathrm{IL}} \varphi$ if $w \Vdash \varphi$ holds in at all worlds w in all Kripke models. Finally, we will write $\Gamma \models_{\mathrm{IL}} \varphi$ if for any world w in any Kripke model (W, R, f, τ) and any assignment, if all formulas in Γ are forced at w under that assignment, the formula φ is forced under that assignment as well.

2. Consistency properties *à la* Beth for intuitionistic predicate logic

We extend the forcing notion to signed formulas as we did for intuitionistic propositional logic: we will write

$$\begin{array}{lll} w \Vdash \mathbf{t}\varphi[\alpha] & \text{if} & w \Vdash \varphi[\alpha] \\ w \Vdash \mathbf{f}\varphi[\alpha] & \text{if} & w \nVdash \varphi[\alpha] \end{array}$$

(where the latter is still not equivalent to $w \Vdash \neg\varphi[\alpha]$!).

But now, besides the special α-formulas, there is a another special case to consider: namely, the case of δ-formulas of the form $\mathbf{f}\,\forall x\,\varphi$, for the reasons we have seen before. These will be called *special*, while others will be called *normal*. Given this observation, I trust the following definition is what one would expect.

Definition 2.1. Let $\mathcal{C}$ be a collection of sets of signed formulae. It will be called a *consistency property* à la *Beth (for intuitionistic predicate logic)*, if for every $\Gamma \in \mathcal{C}$, the following hold:

(1) Γ does not contain both a literal and its dual.
(2) $\mathbf{t}\bot \notin \Gamma$.
(3) if $\sigma \in \Gamma$ and σ is a normal α-formula, then also $\Gamma, \sigma_1, \sigma_2 \in \mathcal{C}$.
(4) if $\sigma \in \Gamma$ and σ is a special α-formula, then also $\Gamma^{\mathbf{t}}, \sigma_1, \sigma_2 \in \mathcal{C}$.
(5) if $\sigma \in \Gamma$ and σ is a β-formula, then $\Gamma, \sigma_1 \in \mathcal{C}$ or $\Gamma, \sigma_1 \in \mathcal{C}$.
(6) if $\sigma \in \Gamma$ and σ is a γ-formula, then also $\Gamma, \sigma(t) \in \mathcal{C}$ for any term t.
(7) if $\sigma \in \Gamma$ and σ is a normal δ-formula, then also $\Gamma, \sigma(a) \in \mathcal{C}$ for any parameter not occurring in Γ.
(8) if $\sigma \in \Gamma$ and σ is a special δ-formula, then also $\Gamma^{\mathbf{t}}, \sigma(a) \in \mathcal{C}$ for any parameter not occurring in Γ.

Theorem 2.2. (Fundamental theorem on consistency properties for intuitionistic predicate logic) *Assume $\mathcal{C}$ is a consistency property for intuitionistic predicate logic* à la *Beth. Then there is a Kripke model (W, R, f, τ) such that for any $\Gamma^* \in \mathcal{C}$ in which infinitely many parameters do not occur there is a world $w \in W$ and an assignment α for $\mathcal{M}_w$ such that all formulas in Γ^* are forced at the node w under the assignment α.*

The proof should contain no surprises.

Definition 2.3. If U is a universe, then a *Hintikka set (for intuitionistic predicate logic)* relative to U consists of a set of signed formulae Γ such that:

(1) all the parameters in Γ come from U.
(2) Γ does not contain both a literal and its dual.
(3) $\mathbf{t}\bot \notin \Gamma$.
(4) if $\sigma \in \Gamma$ and σ is a normal α-formula, then also $\sigma_1, \sigma_2 \in \Gamma$.
(5) if $\sigma \in \Gamma$ and σ is a β-formula, then $\sigma_1 \in \Gamma$ or $\sigma_2 \in \Gamma$.
(6) if $\sigma \in \Gamma$ and σ is a γ-formula, then also $\sigma(t) \in \Gamma$ for any term t built from parameters in U.
(7) if $\sigma \in \Gamma$ and σ is a normal δ-formula, then also $\sigma(a) \in \Gamma$ for a parameter $a \in U$.

Lemma 2.4. *If $\mathcal{C}$ is a consistency property for intuitionistic predicate logic of finite character and $\Gamma \in \mathcal{C}$, then there is a universe U_∞ and Hintikka set Γ_∞ relative to U_∞ with $\Gamma \subseteq \Gamma_\infty$ and $\Gamma_\infty \in \mathcal{C}$.*

PROOF. Straightforward adaption of the proof of Lemma 3.5. □

PROOF. (Of Theorem 2.2.) Without loss of generality we may assume that $\mathcal{C}$ is a consistency property for intuitionistic predicate logic of finite character.

The Kripke model is now constructed as follows:

- the set of worlds W is the set of pairs (U, Γ) where U is a universe and Γ is a Hintikka set relative to U with $\Gamma \in \mathcal{C}$.
- for two such worlds (U_0, Γ_0) and (U_1, Γ_1) we will put $(U_0, \Gamma_0)R(U_1, \Gamma_1)$ if $U_0 \subseteq U_1$ and $\Gamma_0^{\mathbf{t}} \subseteq \Gamma_1^{\mathbf{t}}$.
- the model $\mathcal{M}_{(U,\Gamma)}$ is built in the usual way from (U, Γ): the underlying universe is the set of terms with parameters from U, the interpretation of an n-ary function symbol f is the function sending an n-tuple of terms $(t_1, \ldots, t_n)$ to the term $f(t_1, \ldots, t_n)$, and the interpretation of an n-ary relation symbol R is the collection of those n-tuples of terms $(t_1, \ldots, t_n)$ such that $\mathbf{t}R(t_1, \ldots, t_n)$ belongs to Γ.
- (U_0, Γ_0) and (U_1, Γ_1) are two worlds, then the transition function is simply the inclusion of the set of terms with parameters from U_0 in the larger set of terms with parameters from U_1.

The proof is finished by proving, by induction of the structure of the signed formula σ, that

> if (U, Γ) is a world, $\sigma \in \Gamma$ and α is assignment for $\mathcal{M}_{(U,\Gamma)}$ which is the identity on U, then $(U, \Gamma) \Vdash \sigma[\alpha]$.

This is straightforward, except for the cases of the special α- and δ-formulas, where we need to appeal to Lemma 2.4. We also appeal to that lemma to create a world (U, Γ) with $\Gamma^* \subseteq \Gamma$, so in that world all the formulas in Γ^* will be forced, relative to any assignment which is the identity on U (and does something arbitrary elsewhere). □

CHAPTER 12

Natural Deduction

Both intuitionistic and classical natural deduction are obtained by adding to the systems for propositional logic the following rules for the quantifiers:

5a. If $\mathcal{D}$ is a proof tree with conclusion $\varphi(a)$ and a does not occur in any of the uncanceled assumptions of $\mathcal{D}$, then also

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi(a)\end{array}}{\forall x\, \varphi}$$

is a proof tree with conclusion $\forall x\, \varphi$. (This rule is called $\forall$-*introduction*.)

5b. If $\mathcal{D}$ is a proof tree with conclusion $\forall x\, \varphi$, then also

$$\frac{\begin{array}{c}\mathcal{D}\\ \forall x\, \varphi\end{array}}{\varphi(t)}$$

is a proof tree for any term t. (This rule is called $\forall$-*elimination*.)

6a. If $\mathcal{D}$ is a proof tree with conclusion $\varphi(t)$ for some t, then also

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi(t)\end{array}}{\exists x\, \varphi}$$

is a proof tree with conclusion $\exists x\, \varphi$. (This rule is called $\exists$-*introduction*.)

6b. If $\mathcal{D}_1$ is a proof tree with conclusion $\exists x\, \varphi$, $\mathcal{D}_2$ is a proof tree with conclusion ψ and a is a parameter which does not occur in ψ or in any of the uncanceled assumptions of $\mathcal{D}_2$, except possibly in assumptions of the form $\varphi(a)$, then also

$$\frac{\begin{array}{c}\\ \mathcal{D}_1\\ \exists x\, \varphi\end{array} \qquad \begin{array}{c}[\varphi(a)]\\ \mathcal{D}_2\\ \psi\end{array}}{\psi}$$

is a proof tree, where one may cancel any occurrence of the assumption $\varphi(a)$ in $\mathcal{D}_2$. (This rule is called $\exists$-*elimination*.)

The soundness and completeness of natural deduction can again be proved using the theory of consistency properties.

CHAPTER 13

Hilbert calculus

Recall that the axioms of classical propositional logic are:

K	$\varphi \to (\psi \to \varphi)$
S	$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
	$\varphi \to \varphi \vee \psi$
	$\psi \to \varphi \vee \psi$
	$(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
	$\varphi \wedge \psi \to \varphi$
	$\varphi \wedge \psi \to \psi$
	$\varphi \to (\psi \to (\varphi \wedge \psi))$
DNE	$\neg\neg\varphi \to \varphi$

In addition, there was one rule: the Modus Ponens rule (from φ and $\varphi \to \psi$, infer ψ).

To get a Hilbert-type system for classical predicate logic one adds two additional axioms:

$$\forall x\, \varphi \to \varphi(t),$$
$$\varphi(t) \to \exists x\, \varphi,$$

for any term t. In addition, there will be two more rules: from $\psi \to \varphi(a)$ one may infer $\psi \to \forall x\, \varphi$, provided the parameter a does not occur in φ or ψ; and from $\varphi(a) \to \psi$ one may infer $\exists x\, \varphi \to \psi$ provided the parameter a does not occur in φ or ψ. This means that the relation $\Gamma \vdash \varphi$ is now inductively defined as follows:

- if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
- if φ is a substitution instance of one of the axioms of above, then $\Gamma \vdash \varphi$;
- if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$ (modus ponens).
- if $\Gamma \vdash \psi \to \varphi(a)$ and the parameter a does not occur in φ, ψ or Γ, then $\Gamma \vdash \psi \to \forall x\, \varphi$.
- if $\Gamma \vdash \varphi(a) \to \psi$ and the parameter a does not occur in φ, ψ or Γ, then $\Gamma \vdash \exists x\, \varphi \to \psi$.

The story for intuitionistic predicate logic is the same: here, as in propositional logic, **DNE** is replaced by the ex falso axiom $\bot \to \varphi$, but the axioms and rules for the quantifiers are identical.

In the same way of for propositional logic, one can now prove the Deduction Theorem and the equivalence with natural deduction, both for classical and intuitionistic predicate logic.