CHAPTER 8

Intuitionistic sequent calculus

We have seen that the fundamental theorem for classical consistency properties can be used to give a very quick proof of the completeness of the classical sequent calculus. This is because the defining properties of a classical consistency property directly mirror the rules of the classical sequent calculus. In this chapter we will set up an intuitionistic sequent calculus, using the definition of an intuitionistic consistency property as our guide. In fact, we will give two sequent calculi, one reflecting the definition of an intuitionistic consistency property *à la* Beth and the other reflecting the definition of an intuitionistic consistency property *à la* Gentzen.

1. Intuitionistic sequent calculus *à la* Beth

The intuitionistic sequent calculus *à la* Beth is very close to the classical sequent calculus. We only have to take into account the rôle of the special α-formulas in the definition of an intuitionistic consistency property *à la* Beth.

$$\text{Axioms} \quad \begin{cases} \Gamma, p \Rightarrow \Delta, p \\ \Gamma, \bot \Rightarrow \Delta \end{cases}$$

	Left	Right
$\wedge$	$\dfrac{\Gamma,\alpha_1,\alpha_2\Rightarrow\Delta}{\Gamma,\alpha_1\wedge\alpha_2\Rightarrow\Delta}$	$\dfrac{\Gamma\Rightarrow\beta_1,\Delta \qquad \Gamma\Rightarrow\beta_2,\Delta}{\Gamma\Rightarrow\beta_1\wedge\beta_2,\Delta}$
$\vee$	$\dfrac{\Gamma,\beta_1\Rightarrow\Delta \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\vee\beta_2\Rightarrow\Delta}$	$\dfrac{\Gamma\Rightarrow\alpha_1,\alpha_2,\Delta}{\Gamma\Rightarrow\alpha_1\vee\alpha_2,\Delta}$
$\rightarrow$	$\dfrac{\Gamma\Rightarrow\Delta,\beta_1 \qquad \Gamma,\beta_2\Rightarrow\Delta}{\Gamma,\beta_1\rightarrow\beta_2\Rightarrow\Delta}$	$\dfrac{\Gamma,\alpha_1\Rightarrow\alpha_2}{\Gamma\Rightarrow\alpha_1\rightarrow\alpha_2,\Delta}$

Soundness is easily checked, as is completeness:

THEOREM 1.1. *If $\Gamma \Rightarrow \Delta$ is an intuitionistic tautology, then it is derivable.*

PROOF. It is immediate that

$$\mathcal{C} = \big\{\{\mathbf{t}\gamma_1,\dots,\mathbf{t}\gamma_n,\mathbf{f}\delta_1,\dots,\mathbf{f}\delta_m\} : \gamma_1,\dots,\gamma_n \Rightarrow \delta_1,\dots,\delta_m \text{ is not derivable}\big\}$$

defines an intuitionistic consistency property *à la* Beth. So if $\gamma_1,\dots,\gamma_n \Rightarrow \delta_1,\dots,\delta_m$ is not derivable, $\{\mathbf{t}\gamma_1,\dots,\mathbf{t}\gamma_n,\mathbf{f}\delta_1,\dots,\mathbf{f}\delta_m\}$ belongs to $\mathcal{C}$. The fundamental theorem on consistency properties then tells us that there is a world in some Kripke model where these formulas are forced, showing that $\gamma_1,\dots,\gamma_n \Rightarrow \delta_1,\dots,\delta_m$ is not a tautology. □

Proofs of the following lemmata are variations on things we have seen before and are therefore omitted.

LEMMA 1.2. (Subformula property) *If π is a derivation in the intuitionistic sequent calculus* à la *Beth with endsequent σ, then every formula occurring in π is a subformula of a formula occurring in σ.*

LEMMA 1.3. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a derivation π in the intuitionistic sequent calculus* à la *Beth, and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is derivable as well.*

LEMMA 1.4. (Inversion Lemma) *The rules for introducing $\wedge$ and $\vee$ on the left and right in the intuitionistic sequent calculus* à la *Beth are invertible: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by a rule introducing a disjunction or conjunction, then there are derivations π_i of the σ_i as well. For $\to$-introduction on the left we only have that from a derivation of $\Gamma, \varphi \to \psi \Rightarrow \Delta$ we can effectively find a derivation of $\Gamma, \psi \Rightarrow \Delta$, while for $\to$-introduction on the right we have that from a derivation of $\Gamma \Rightarrow \Delta, \varphi \to \psi$ we can effectively find a derivation of $\Gamma, \varphi \Rightarrow \psi, \Delta$.*

2. Intuitionistic sequent calculus *à la* Gentzen

We will now formulate a version of the intuitionistic sequent calculus using intuitionistic consistency properties *à la* Gentzen as our guide. But there is a twist: what we will exploit is that it is possible to have an intuitionistic consistency property *à la* Gentzen $\mathcal{C}$ such that each $\Gamma \in \mathcal{C}$ contains only one formula signed with an $\mathbf{f}$. This allows us to formulate a version of the intuitionistic sequent calculus in which in each sequent there occurs only one formula on the right of the arrow $\Rightarrow$. Indeed, this is what Gentzen did in his initial investigations. (The sequent calculus that we discussed in the previous section is a lesser known variant deriving from the semantic investigations of Beth and others.) So define an *intuitionistic sequent* to be an expression of the form $\Gamma \Rightarrow \varphi$ where Γ is a finite set of formulas. It is valid, consistent, et cetera, if $\bigwedge \Gamma \to \varphi$ holds.

All this leads to the following axioms and rules:

$$\text{Axioms} \quad \begin{cases} \Gamma, p \Rightarrow p \\ \Gamma, \bot \Rightarrow \varphi \end{cases}$$

	Left	Right
$\wedge$	$\dfrac{\Gamma,\alpha_1,\alpha_2 \Rightarrow \varphi}{\Gamma,\alpha_1 \wedge \alpha_2 \Rightarrow \varphi}$	$\dfrac{\Gamma \Rightarrow \beta_1 \qquad \Gamma \Rightarrow \beta_2}{\Gamma \Rightarrow \beta_1 \wedge \beta_2}$
$\vee$	$\dfrac{\Gamma,\beta_1 \Rightarrow \varphi \qquad \Gamma,\beta_2 \Rightarrow \varphi}{\Gamma,\beta_1 \vee \beta_2 \Rightarrow \varphi}$	$\dfrac{\Gamma \Rightarrow \alpha_1}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2} \qquad \dfrac{\Gamma \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}$
$\to$	$\dfrac{\Gamma \Rightarrow \beta_1 \qquad \Gamma,\beta_2 \Rightarrow \varphi}{\Gamma,\beta_1 \to \beta_2 \Rightarrow \varphi}$	$\dfrac{\Gamma,\alpha_1 \Rightarrow \alpha_2}{\Gamma \Rightarrow \alpha_1 \to \alpha_2}$

One readily checks soundness. In addition, one has:

THEOREM 2.1. *If $\Gamma \Rightarrow \varphi$ is an intuitionistic tautology, then it is derivable in the intuitionistic sequent calculus* à la *Gentzen.*

PROOF. It is immediate that

$$\mathcal{C} = \big\{\{\mathbf{t}\gamma_1, \ldots, \mathbf{t}\gamma_n, \mathbf{f}\varphi\} : \gamma_1, \ldots, \gamma_n \Rightarrow \varphi \text{ is not derivable}\big\}$$

defines an intuitionistic consistency property *à la* Gentzen. □

We also have:

LEMMA 2.2. (Subformula property) *If π is a derivation in the intuitionistic sequent calculus* à la *Gentzen with endsequent σ, then every formula occurring in π is a subformula of a formula occurring in σ.*

LEMMA 2.3. (Weakening) *If $\Gamma \Rightarrow \varphi$ is the endsequent of a derivation π in the intuitionistic sequent calculus* à la *Gentzen and $\Gamma \subseteq \Gamma'$, then $\Gamma' \Rightarrow \varphi$ is derivable as well.*

LEMMA 2.4. (Inversion Lemma) *The rules for introducing conjunctions on the left and right, implications on the right and disjunctions on the left are invertible in the intuitionistic sequent calculus* à la *Gentzen: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one of these rules, then there are derivations π_i of the σ_i as well. For $\to$-introduction on the left we only have that from a derivation of $\Gamma, \varphi \to \psi \Rightarrow \chi$ we can find a derivation of $\Gamma, \psi \Rightarrow \chi$, while $\vee$-introduction on the right is simply not invertible.*