CHAPTER 7

Intuitionistic consistency properties

The aim of this chapter is to adapt what we did in the previous chapter to the intuitionistic case. Of course, this means that instead of classical models we will work with Kripke models for intuitionistic logic. The notion of signed formula will be the same, but we will have to define what it means for a signed formula to be forced at a world w in some Kripke model (W, R, f). We do this as follows: we define $w \Vdash \mathbf{t}\varphi$ to be $w \Vdash \varphi$ and $w \Vdash \mathbf{f}\varphi$ to be $w \nVdash \varphi$ (note that this is *not* equivalent to $w \Vdash \neg\varphi$!).

We will again formulate a notion of an intuitionistic consistency property and prove a fundamental theorem, showing that for any set of signed formulae belonging to some consistency property there is a world w in some Kripke model (W, R, f) at which all formulas in Γ must be forced. In fact, we will formulate two intuitionistic consistency properties, one *à la* Beth and one *à la* Gentzen.

1. Consistency properties *à la* Beth

Consistency properties *à la* Beth are closest to the classical ones. Indeed, much is as before but there is one big difference, having to do with the α-formulas of the form $\mathbf{f}(\varphi \to \psi)$: these will be called *special α-formulas*, the others will be called *normal*. The distinguishing feature of these special α-formulas is that their validity refers to what happens in some later world, which need not be the world we are in now. And what we know (from our present point of view) is that the things that are true now, will still be true there, but not much else. This is what is reflected in the following definition, where we have used the following notation:

$$\Gamma^{\mathbf{t}} = \{\mathbf{t}\varphi \colon \mathbf{t}\varphi \in \Gamma\}.$$

Definition 1.1. Let $\mathcal{C}$ be a non-empty collection of sets of signed formulae. $\mathcal{C}$ will be called a *consistency property* à la *Beth for intuitionistic propositional logic* if for any $\Gamma \in \mathcal{C}$, we have:

(1) Γ does not contain a literal and its dual.
(2) $\mathbf{t}\bot \notin \Gamma$.
(3) if $\sigma \in \Gamma$ and σ is a normal α-formula, then also $\Gamma, \sigma_1, \sigma_2 \in \mathcal{C}$.
(4) if $\sigma \in \Gamma$ and σ is a special α-formula, then also $\Gamma^{\mathbf{t}}, \sigma_1, \sigma_2 \in \mathcal{C}$.
(5) if $\sigma \in \Gamma$ and σ is a β-formula, then $\Gamma, \sigma_1 \in \mathcal{C}$ or $\Gamma, \sigma_2 \in \mathcal{C}$.

Note that it is still the case that any consistency property can be extended to one of finite character. We also need an intuitionistic version of a Hintikka set: what happens here is that there is *no* clause for special α-formulas.

Definition 1.2. A *Hintikka set (*à la *Beth for intuitionistic propositional logic)* is a set of signed formulae Γ satisfying the following properties:

(1) Γ does not contain a literal and its dual.
(2) $\mathbf{t}\bot \not\in \Gamma$.
(3) if $\sigma \in \Gamma$ and σ is a normal α-formula, then $\sigma_1, \sigma_2 \in \Gamma$.
(4) if $\sigma \in \Gamma$ and σ is a β-formula, then $\sigma_1 \in \Gamma$ or $\sigma_2 \in \Gamma$.

We again have:

LEMMA 1.3. *If $\mathcal{C}$ is a consistency property for intuitionistic propositional logic of finite character, then any $\Gamma \in \mathcal{C}$ can be extended to an intuitionistic Hintikka set which still belongs to $\mathcal{C}$.*

PROOF. Same proof method as before works (working with an enumeration of the normal α-formulas and the β-formulas in which each such formula is repeated infinitely often). □

THEOREM 1.4. (Fundamental theorem on consistency properties) *Let $\mathcal{C}$ be a consistency property for intuitionistic propositional logic* à la *Beth. Then there is a Kripke model (W, R, f) such that for any $\Gamma_0 \in \mathcal{C}$ there is a world $w \in W$ such that all formulas in Γ_0 are forced at w.*

PROOF. Without loss of generality we may assume that $\mathcal{C}$ is a consistency property for intuitionistic propositional logic of finite character.

The Kripke model is now constructed as follows. Let W be the collection of Hintikka sets belonging to $\mathcal{C}$. If $\Gamma \in W$, we put $f(\Gamma) = \{p \in P : \mathbf{t}p \in \Gamma\}$ and for $\Gamma, \Delta \in W$, we put $\Gamma R \Delta$ if $\Gamma^{\mathbf{t}} \subseteq \Delta^{\mathbf{t}}$.

Now we prove by induction on the structure of the signed formula σ:

if $\Gamma \in W$ and $\sigma \in \Gamma$, then $\Gamma \Vdash \sigma$.

This is all fairly straightforward, except for the case where σ is a special formula of the form $\mathbf{f}(\varphi \to \psi)$. If $\Gamma \in W$, then in particular $\Gamma \in \mathcal{C}$, so if $\sigma \in \Gamma$, then $\Gamma^{\mathbf{t}}, \mathbf{t}\varphi, \mathbf{f}\psi \in \mathcal{C}$ as well. Using the previous lemma we can extend this set $\Gamma^{\mathbf{t}}, \mathbf{t}\varphi, \mathbf{f}\psi \in \mathcal{C}$ to a Hintikka set Δ which still belongs to $\mathcal{C}$. So $\Delta \in W$ and $\Gamma R \Delta$. The induction hypothesis tells us that φ holds in Δ, while ψ does not, so we have what we want.

To finish the proof then, we can use the previous lemma again to find a Hintikka set Γ in $\mathcal{C}$ which extends Γ_0. So $\Gamma \in W$ and all formulas in Γ will be forced at Γ: this applies in particular to all the formulas in Γ_0. □

2. Consistency properties *à la* Gentzen

Probably the best way to think about consistency properties *à la* Gentzen is as a refinement of consistency properties *à la* Beth. The starting point is the observation that where the validity of formulas signed with $\mathbf{t}$ is preserved along the relation R (indeed, that is persistence), validity of formulas signed with $\mathbf{f}$ is reflected backwards along the relation R. So if you want to construct a Kripke model and validate a formula like $\mathbf{f}(\alpha_1 \vee \alpha_2)$ at some world, it suffices to make sure that both $\mathbf{f}\alpha_1$ and $\mathbf{f}\alpha_2$ are valid at some later worlds; given that the accessibility relation in the proof is defined by saying that $\Gamma R \Delta$ holds if $\Gamma^{\mathbf{t}} \subseteq \Delta^{\mathbf{t}}$, it stands to reason that the proof above would still work with the following weaker notion of consistency property.

DEFINITION 2.1. Let $\mathcal{C}$ be a non-empty collection of sets of signed formulae. $\mathcal{C}$ will be called a *consistency property* à la *Gentzen (for intuitionistic propositional logic)* if for any $\Gamma \in \mathcal{C}$, we have:

(1) Γ does not contain a literal and its dual.
(2) $\mathbf{t}\bot \notin \Gamma$.
(3) if $\sigma \in \Gamma$ and $\sigma = \mathbf{t}(\alpha_1 \wedge \alpha_2)$, then also $\Gamma, \mathbf{t}\alpha_1, \mathbf{t}\alpha_2 \in \mathcal{C}$.
(4) if $\sigma \in \Gamma$ and $\sigma = \mathbf{f}(\alpha_1 \vee \alpha_2)$, then also $\Gamma^{\mathbf{t}}, \mathbf{f}\alpha_1 \in \mathcal{C}$ and $\Gamma^{\mathbf{t}}, \mathbf{f}\alpha_2 \in \mathcal{C}$.
(5) if $\sigma \in \Gamma$ and $\sigma = \mathbf{f}(\alpha_1 \to \alpha_2)$, then also $\Gamma^{\mathbf{t}}, \mathbf{t}\alpha_1, \mathbf{f}\alpha_2 \in \mathcal{C}$.
(6) if $\sigma \in \Gamma$ and $\sigma = \mathbf{f}(\beta_1 \wedge \beta_2)$, then $\Gamma^{\mathbf{t}}, \mathbf{f}\beta_1 \in \mathcal{C}$ or $\Gamma^{\mathbf{t}}, \mathbf{f}\beta_2 \in \mathcal{C}$.
(7) if $\sigma \in \Gamma$ and $\sigma = \mathbf{t}(\beta_1 \vee \beta_2)$, then $\Gamma, \mathbf{t}\beta_1 \in \mathcal{C}$ or $\Gamma, \mathbf{t}\beta_2 \in \mathcal{C}$.
(8) if $\sigma \in \Gamma$ and $\sigma = \mathbf{t}(\beta_1 \to \beta_2)$, then $\Gamma^{\mathbf{t}}, \mathbf{f}\beta_1 \in \mathcal{C}$ or $\Gamma, \mathbf{t}\beta_2 \in \mathcal{C}$.

We also have to formulate a suitable notion of a Hintikka set. The main difference is that the notion of Hintikka set is now defined relative to a consistency property.

DEFINITION 2.2. Let $\mathcal{C}$ be a consistency property *à la* Gentzen for intuitionistic propositional logic. A set of signed formulae Γ is a *Hintikka set for* $\mathcal{C}$, if

(1) $\Gamma \in \mathcal{C}$, and
(2) if $\Gamma, \mathbf{t}\varphi \in \mathcal{C}$, then $\mathbf{t}\varphi \in \Gamma$.

LEMMA 2.3. *Let $\mathcal{C}$ be a consistency property of finite character* à la *Gentzen.*

(1) *Any $\Gamma \in \mathcal{C}$ can be extended to a Hintikka set for $\mathcal{C}$.*
(2) *If Γ is a Hintikka set for $\mathcal{C}$ and $\mathbf{t}(\alpha_1 \wedge \alpha_2) \in \Gamma$, then $\mathbf{t}\alpha_1, \mathbf{t}\alpha_2 \in \Gamma$.*
(3) *If Γ is a Hintikka set for $\mathcal{C}$ and $\mathbf{t}(\beta_1 \vee \beta_2) \in \Gamma$, then $\mathbf{t}\beta_1 \in \Gamma$ or $\mathbf{t}\beta_2 \in \Gamma$.*

PROOF. Part (1) is a simple variation on things that we have seen before; the other two parts are left as an easy exercise. □

THEOREM 2.4. (Fundamental theorem on consistency properties for intuitionistic logic *à la* Gentzen) *Let $\mathcal{C}$ be a consistency property for intuitionistic logic à la Gentzen. Then there is a Kripke model (W, R, f) such that for any $\Gamma_0 \in \mathcal{C}$ there is a world $w \in W$ such that all formulas in Γ_0 are forced at w.*

PROOF. Again, without loss of generality we may assume that $\mathcal{C}$ is a consistency property of finite character.

The Kripke model is now constructed as follows. Let W be the collection of Hintikka sets for $\mathcal{C}$. If $\Gamma \in W$, we put $f(\Gamma) = \{ p \colon \mathbf{t}p \in \Gamma \}$ and for $\Gamma, \Delta \in W$ we put $\Gamma R \Delta$ if $\Gamma^{\mathbf{t}} \subseteq \Delta^{\mathbf{t}}$. The crucial property that we need is that for any signed formula σ, we have

if $\Gamma \in W$ and $\sigma \in \Gamma$, then $\Gamma \Vdash \sigma$.

We prove this by induction on the structure of σ: the only case which deserves special treatment is the case where $\sigma = \mathbf{t}(\beta_1 \to \beta_2)$. So suppose $\Gamma, \Delta \in W$, $\mathbf{t}(\beta_1 \to \beta_2) \in \Gamma$, $\Gamma R \Delta$ and $\Delta \Vdash \beta_1$. We need to prove $\Delta \Vdash \beta_2$. First observe that since $\Gamma R \Delta$, we also have $\mathbf{t}(\beta_1 \to \beta_2) \in \Delta$. So either $\Delta^{\mathbf{t}}, \mathbf{f}\beta_1 \in \mathcal{C}$ or $\Delta, \mathbf{t}\beta_2 \in \mathcal{C}$.

If $\Delta^{\mathbf{t}}, \mathbf{f}\beta_1 \in \mathcal{C}$, then by the previous lemma $\Delta^{\mathbf{t}}, \mathbf{f}\beta_1$ can be extended to a Hintikka set Δ'. But then $\Delta' \in W, \Delta R \Delta'$ and $\mathbf{f}\beta_1 \in \Delta'$. By induction hypothesis the latter means $\Delta' \nVdash \beta_1$. Hence by monotonicity we obtain $\Delta \nVdash \beta_1$, which is a contradiction.

So we obtain $\Delta, \mathbf{t}\beta_2 \in \mathcal{C}$ and hence $\mathbf{t}\beta_2 \in \Delta$, since Δ is a Hintikka set for $\mathcal{C}$. So the induction hypothesis yields $\Delta \Vdash \beta_2$, as desired.

The other cases are a lot easier and omitted. □