CHAPTER 5

Classical sequent calculus

One of the most important proof systems is the *sequent calculus*, which, like natural deduction, was invented by the German proof-theorist Gerhard Gentzen. Sequent calculus also resembles natural deduction in that the proofs look like trees. The main difference, however, is that in the sequent calculus the nodes in the trees are labeled with *sequents*, not formulas. A sequent is an expression of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas; its intuitive meaning is $\bigwedge \Gamma \to \bigvee \Delta$, that is, the conjunction of all the formulas in Γ implies the disjunction of all the formulas in Δ. In particular, a sequent $\Gamma \Rightarrow \Delta$ is a tautology, or consistent, or ..., precisely when $\bigwedge \Gamma \to \bigvee \Delta$ is.

1. The rules of the sequent calculus for classical propositional logic

In this chapter we will only look at the *classical* sequent calculus. There are many different variations of the classical sequent calculus, but in these notes we will only consider one variant. This variant has two axioms and for each logical connective two inference rules, one introducing it on the left and one introducing it on the right:

$$\text{Axioms} \quad \begin{cases} \Gamma, p \Rightarrow \Delta, p \\ \Gamma, \bot \Rightarrow \Delta \end{cases}$$

	Left	Right
$\wedge$	$\frac{\Gamma,\alpha_1,\alpha_2 \Rightarrow \Delta}{\Gamma,\alpha_1 \wedge \alpha_2 \Rightarrow \Delta}$	$\frac{\Gamma \Rightarrow \beta_1,\Delta \qquad \Gamma \Rightarrow \beta_2,\Delta}{\Gamma \Rightarrow \beta_1 \wedge \beta_2,\Delta}$
$\vee$	$\frac{\Gamma,\beta_1 \Rightarrow \Delta \qquad \Gamma,\beta_2 \Rightarrow \Delta}{\Gamma,\beta_1 \vee \beta_2 \Rightarrow \Delta}$	$\frac{\Gamma \Rightarrow \alpha_1,\alpha_2,\Delta}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2,\Delta}$
$\to$	$\frac{\Gamma \Rightarrow \Delta,\beta_1 \qquad \Gamma,\beta_2 \Rightarrow \Delta}{\Gamma,\beta_1 \to \beta_2 \Rightarrow \Delta}$	$\frac{\Gamma,\alpha_1 \Rightarrow \alpha_2,\Delta}{\Gamma \Rightarrow \alpha_1 \to \alpha_2,\Delta}$

A sequent which appears as a conclusion of a derivation π is often called the *endsequent* of the derivation.

THEOREM 1.1. (Soundness of the sequent calculus) *If* $\Gamma \Rightarrow \Delta$ *is derivable in the classical sequent calculus, then it is a tautology.*

The converse (completeness) holds as well and will be proved in the next chapter.

2. Properties of the sequent calculus for classical propositional logic

The sequent calculus has many remarkable properties. The most obvious is that at each step a new formula is introduced by combining ingredients that were already present. For this reason derivations in the sequent calculus obey the *subformula property.*

LEMMA 2.1. (Subformula property) *If π is a derivation with endsequent σ, then every formula occurring in π is a subformula of a formula occurring in σ.*

PROOF. By direct inspection of the rules. (Alternatively: by induction on the derivation π.) □

The following is also quite easy to see:

LEMMA 2.2. (Weakening) *If $\Gamma \Rightarrow \Delta$ is the endsequent of a derivation π and $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, then $\Gamma' \Rightarrow \Delta'$ is derivable as well.*

PROOF. Take the derivation π and add all the formulas from Γ' to the left of the arrow $\Rightarrow$ and all the formulas from Δ' to the right of the arrow $\Rightarrow$ in every sequent in π. The result is still a derivation and it has endsequent $\Gamma' \Rightarrow \Delta'$. (Alternatively: by induction on the derivation π.) □

The following lemma states a far less obvious property. It is proved by reasoning backwards: that is, given a sequent σ it investigates how it could be derived by searching through all possible inferences which have that sequent as its conclusion. Clearly, this can be iterated: one can then proceed to study all possible ways of obtaining any sequent from which σ can be obtained and so on. Since this is the main proof method in arguing about derivations in the sequent calculus and it is quite easy to make mistakes when trying to applying it, we take some time to point out the pitfalls.

First of all, it is easy to believe that the only way to obtain

$$p \wedge q \Rightarrow p$$

is from

$$p, q \Rightarrow p$$

by applying the L$\wedge$-rule. That. however, is not true, as it could also have been obtained from

$$p, q, p \wedge q \Rightarrow p$$

(also by the L$\wedge$-rule): the reason for this is that also this inference step is an instance of

$$\frac{\Gamma, p, q \;\Rightarrow\; p}{\Gamma, p \wedge q \;\Rightarrow\; p}$$

but with $\Gamma = \{p \wedge q\}$ instead of $\Gamma = \emptyset$. (Remember that on the left and right of the arrow we have sets!)

Note that the weakening lemma implies that whenever, for example, $\Gamma, \varphi, \psi \Rightarrow \Delta$ is derivable then so is $\Gamma, \varphi \wedge \psi, \varphi, \psi \Rightarrow \Delta$. This has the following slightly paradoxical consequence: when reasoning backwards we always have to consider the possibility that the formula we are "building" was already present in *all* the premises; and, what is more, we may reduce, without loss of generality, the situation where this does not happen to the case where it does.

To make things more complicated: as it stands it is not even clear at each inference step which rule has been applied; for example, the following

$$\frac{p, q, p \wedge q \ \Rightarrow \ r, s, r \vee s}{p, q, p \wedge q \ \Rightarrow \ r, s, r \vee s}$$

is both an application of $\wedge$-introduction on the left and $\vee$-introduction on the right! This phenomenon, however, is a nuisance and with a bit of effort one can see that this phenomenon can only occur in case the conclusion of an inference step coincides with one of its premises. So we will typically assume that derivations are *sensible* in that at each inference step the conclusion is distinct from all of its premises: it is clear that each derivation that is not sensible can be transformed in a shorter one that is sensible.

For inference steps in a sensible derivation it is always clear which rule has been applied and also to which formulas (because in such inference steps there is always a single new formula appearing in the conclusion or some ingredients which have been absorbed into a particular formula). In that case we divide the formulas in the premise(s) into *active* and *passive* formulas, where in the rules above $\alpha_1, \alpha_2, \beta_1, \beta_2$ are active, while the others are passive; similarly, we divide the formulas in the conclusion into two groups, where $\alpha_1 \Box \alpha_2$ and $\beta_1 \Box \beta_2$ are *principal formulas* and the others are *side formulas.*

LEMMA 2.3. (Inversion Lemma) *Each of the rules in the classical sequent calculus is* invertible*: if there is a derivation π of a sequent σ and σ can be obtained from sequents $\sigma_1, \ldots, \sigma_n$ by one the rules, then there are derivations π_i of the σ_i as well.*

PROOF. This is again a proof by induction on the derivation π. We will assume that π is sensible.

There are many case to consider, so we will only discuss one illustrative case and leave the others to the reader (in case he or she is bored). Suppose the conclusion of π is $\Gamma, \varphi \wedge \psi \Rightarrow \Delta$ and we want to argue that this means that we must also have a derivation of $\Gamma, \varphi, \psi \Rightarrow \Delta$.

First we must consider the case that $\Gamma, \varphi \wedge \psi \Rightarrow \Delta$ is axiom, which means either that both Γ and Δ share a propositional variable p or that Γ contains $\bot$. In both cases also $\Gamma, \varphi, \psi \Rightarrow \Delta$ is an axiom.

Let us now the consider the case where $\varphi \wedge \psi$ is principal in the last inference in π, which means that it has been obtained in π from either $\Gamma, \varphi, \psi \Rightarrow \Delta$ or $\Gamma, \varphi \wedge \psi, \varphi, \psi \Rightarrow \Delta$. In the first case we are done immediately; in the second case we can apply the induction hypothesis on the smaller derivation resulting in $\Gamma, \varphi \wedge \psi, \varphi, \psi \Rightarrow \Delta$ (that is, the derivation π minus the last step) to deduce that $\Gamma, \varphi, \psi \Rightarrow \Delta$ is derivable.

Finally, we consider the case where $\varphi \wedge \psi$ is a side formula in the last inference in π. There are many ways in which this could happen, but let us just consider the case where it is obtained from $\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \beta_1$ and $\Gamma, \varphi \wedge \psi, \beta_2 \Rightarrow \Delta$ by applying rule introducing $\to$ on the left (so $\beta_1 \to \beta_2 \in \Gamma$). By applying the induction hypothesis on the derivations of $\Gamma, \varphi \wedge \psi \Rightarrow \beta_1$ and $\Gamma, \varphi \wedge \psi, \beta_2 \Rightarrow \Delta$ we obtain derivations of $\Gamma, \varphi, \psi \Rightarrow \Delta, \beta_1$ and $\Gamma, \varphi, \psi, \beta_2 \Rightarrow \Delta$. Taking those derivations and applying the rule introducing $\to$ on the left gives us a derivation of $\Gamma, \varphi, \psi \Rightarrow \Delta$ (as $\beta_1 \to \beta_2 \in \Gamma$) and that is precisely what we want. $\Box$

The upshot of these lemmas is that the sequent calculus is very suitable for *backwards proof search*: starting from a sequent σ one can systematically search through all possible ways of deriving it. In fact, the inversion lemma tells us that we can always pick one possible way of deriving the sequent and only investigate that possibility. Clearly, some possibilities are more attractive to explore than others: in particular, one wants to avoid branching. When there are no more rules which can be applied, then the sequent is either an axiom or it is not. If all branches end with an axiom, then the formula must be derivable; if one branch ends with something which is not axiom, then the sequent is not derivable. This gives one a decision method for classical propositional logic (different from truth tables).

CHAPTER 6

Completeness proofs

In this chapter we prove the completeness of the classical sequent calculus we introduced in the previous chapter. In fact, this will be an immediate consequence of a general model existence theorem that we will prove based on the notion of a *consistency property.* We haven chosen this method because it can also be used to show completeness of classical natural deduction as well as many other things (such as the compactness theorem). But first we will introduce a technical device (signed formulas) which will allow us to save some ink.

Note: Throughout this chapter we will only consider classical models $\mathcal{M}$.

1. Signed formulas

DEFINITION 1.1. A *signed formula* is an expression of the form $\mathbf{t}\varphi$ or $\mathbf{f}\varphi$ where φ is a propositional formula.

We extend the notion of validity to signed formulas as follows: $\mathcal{M} \models \mathbf{t}\varphi$ if $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \mathbf{f}\varphi$ if $\mathcal{M} \not\models \varphi$ (which is of course equivalent to $\mathcal{M} \models \neg\varphi$). The formulas $\mathbf{t}\varphi$ and $\mathbf{f}\varphi$ are called each other's duals, and the formulas of the form $\mathbf{t}p$ and $\mathbf{f}p$, where p is a propositional variable, are called *literals.*

DEFINITION 1.2. Signed formulas of the form $\mathbf{t}(\varphi \wedge \psi)$, $\mathbf{f}(\varphi \vee \psi)$ and $\mathbf{f}(\varphi \to \psi)$ are called α*-formulas.* If α is an α-formula, then α_1 and α_2 are the signed formulas given by the following table:

α	α_1	α_2
$\mathbf{t}(\varphi \wedge \psi)$	$\mathbf{t}\varphi$	$\mathbf{t}\psi$
$\mathbf{f}(\varphi \vee \psi)$	$\mathbf{f}\varphi$	$\mathbf{f}\psi$
$\mathbf{f}(\varphi \to \psi)$	$\mathbf{t}\varphi$	$\mathbf{f}\psi$

The signed formulas of the form $\mathbf{f}(\varphi \wedge \psi)$, $\mathbf{t}(\varphi \vee \psi)$, $\mathbf{t}(\varphi \to \psi)$ are called β*-formulas.* If β is a β-formula, then β_1 and β_2 are the signed formulas given by the following table:

β	β_1	β_2
$\mathbf{f}(\varphi \wedge \psi)$	$\mathbf{f}\varphi$	$\mathbf{f}\psi$
$\mathbf{t}(\varphi \vee \psi)$	$\mathbf{t}\varphi$	$\mathbf{t}\psi$
$\mathbf{t}(\varphi \to \psi)$	$\mathbf{f}\varphi$	$\mathbf{t}\psi$

2. Consistency properties for classical propositional logic

The aim of this section is to prove a general result guaranteeing the existence of models with certain prescribed properties. To state it we need the following notion.

Definition 2.1. Let $\mathcal{C}$ be a collection of sets of signed formulas. $\mathcal{C}$ will be called a *consistency property (for classical propositional logic)* if for any set $\Gamma \in \mathcal{C}$ we have:

(1) Γ does not contain a literal and its dual.
(2) $\mathbf{t}\bot \notin \Gamma$.
(3) if $\sigma \in \Gamma$ and σ is an α-formula, then also $\Gamma, \sigma_1, \sigma_2 \in \mathcal{C}$.
(4) if $\sigma \in \Gamma$ and σ is a β-formula, then $\Gamma, \sigma_1 \in \mathcal{C}$ or $\Gamma, \sigma_2 \in \mathcal{C}$.

Theorem 2.2. (Fundamental theorem on consistency properties for classical propositional logic) *If $\mathcal{C}$ is a consistency property and $\Gamma \in \mathcal{C}$, then there is a model $\mathcal{M}$ of classical propositional logic such that all formulas in Γ hold in $\mathcal{M}$.*

This theorem will follow from a sequence of lemmas. But let us first observe that the completeness of the classical sequent calculus is an immediate consequence.

Corollary 2.3. (Completeness of the classical sequent calculus) *If $\Gamma \Rightarrow \Delta$ is a tautology in classical propositional logic, then it is derivable in the classical sequent calculus.*

Proof. First of all, note that

$$\mathcal{C} = \big\{\{\mathbf{t}\gamma \colon \gamma \in \Gamma\} \cup \{\mathbf{f}\delta \colon \delta \in \Delta\} \colon \Gamma \Rightarrow \Delta \text{ is not derivable } \big\}$$

defines a consistency property. Now we argue by contraposition: if $\Gamma \Rightarrow \Delta$ is not derivable, then we have

$$\{\mathbf{t}\gamma \colon \gamma \in \Gamma\} \cup \{\mathbf{f}\delta \colon \delta \in \Delta\} \in \mathcal{C},$$

by definition of $\mathcal{C}$. Therefore there is a model in which all the formulas in Γ are true and all the formulas in Δ are false by the previous theorem. So $\Gamma \Rightarrow \Delta$ is a not a tautology. $\square$

We now proceed to prove Theorem 2.2. First a definition.

Definition 2.4. A consistency property $\mathcal{C}$ is of *finite character* in case $\Gamma \in \mathcal{C}$ holds precisely if $\Gamma_0 \in \mathcal{C}$ for any finite subset $\Gamma_0 \subseteq \Gamma$.

Lemma 2.5. *Any consistency property can be extended to one of finite character.*

Proof. If $\mathcal{C}$ is a consistency property, then so are:

$$\begin{array}{rcl} \mathcal{C}_1 & = & \{\Gamma \colon (\exists \Gamma')\, \Gamma \subseteq \Gamma' \text{ and } \Gamma' \in \mathcal{C}\}, \\ \mathcal{C}_2 & = & \{\Gamma \colon \text{all finite subsets } \Gamma_0 \subseteq \Gamma \text{ belong to } \mathcal{C}_1\}. \end{array}$$

Then $\mathcal{C}_2$ is of finite character and $\mathcal{C} \subseteq \mathcal{C}_2$. $\square$

Definition 2.6. A collection Γ of signed formulas is called a *Hintikka set (for classical propositional logic)*, if:

(1) Γ does not contain a literal and its dual.
(2) $\mathbf{t}\bot \notin \Gamma$.
(3) if $\sigma \in \Gamma$ and σ is an α-formula, then $\sigma_1, \sigma_2 \in \Gamma$.
(4) if $\sigma \in \Gamma$ and σ is a β-formula, then $\sigma_1 \in \Gamma$ or $\sigma_2 \in \Gamma$.

LEMMA 2.7. *If $\mathcal{C}$ is a consistency property for classical propositional logic of finite character and $\Gamma \in \mathcal{C}$, then Γ can be extended to a Hintikka set Γ' which still belongs to $\mathcal{C}$.*

PROOF. The idea is to create an increasing sequence of sets of signed formulae $\Gamma_n \in \mathcal{C}$ with $\Gamma_0 = \Gamma$ and $\Gamma' = \bigcup \Gamma_n$ a Hintikka set. Since $\mathcal{C}$ is of a finite character, we will have $\Gamma' \in \mathcal{C}$, as desired.

Obviously, we start by putting $\Gamma_0 = \Gamma$. Now fix an enumeration σ_n of the signed formulae of type α or β in which each such formula occurs infinitely often.

We define Γ_{n+1} once Γ_n has been defined, as follows:

(1) if $\sigma_n \notin \Gamma_n$, then $\Gamma_{n+1} = \Gamma_n$.
(2) if $\sigma_n \in \Gamma_n$ and σ_n is of α-type, then $\Gamma_{n+1} = \Gamma_n, (\sigma_n)_1, (\sigma_n)_2$.
(3) if $\sigma_n \in \Gamma_n$ and σ_n is of β-type and $\Gamma_n, (\sigma_n)_1 \in \mathcal{C}$, then $\Gamma_{n+1} = \Gamma_n, (\sigma_n)_1$.
(4) if $\sigma_n \in \Gamma_n$ and σ_n is of β-type and $\Gamma_n, (\sigma_n)_1 \notin \mathcal{C}$, then $\Gamma_{n+1} = \Gamma_n, (\sigma_n)_2$.

One readily checks that this sequence has the required properties. □

LEMMA 2.8. *Let Γ be a Hintikka set for classical propositional logic. Then Γ has a model $\mathcal{M}$.*

PROOF. Define a model as follows: $\mathcal{M} = \{p \in P \colon \mathbf{t}p \in \Gamma\}$. Now one easily proves by induction on the structure of the signed formula σ, that $\sigma \in \Gamma$ implies $\mathcal{M} \models \sigma$. So $\mathcal{M}$ is a model of Γ. □

Together the previous three lemmas imply the theorem: if $\mathcal{C}$ is a consistency property and $\Gamma \in \mathcal{C}$, then one first extends $\mathcal{C}$ to a consistency property of finite character $\mathcal{C}'$. Then also $\Gamma \in \mathcal{C}'$, so by the second lemma Γ can be enlarged to a Hintikka set Γ' which still belongs to $\mathcal{C}'$. But then the third lemma gives one a model $\mathcal{M}$ of Γ'; and since $\Gamma \subseteq \Gamma'$, the model $\mathcal{M}$ is also a model of Γ.