CHAPTER 3

Hilbert-style proof calculus

Natural deduction is arguably the nicest proof calculus around, but it is certainly not the oldest or the simplest. In fact, the simplest kind of proof calculi that exist may be the Hilbert-style proof calculi (sometimes also called Frege-style proof calculi); and despite the fact that they are the oldest systems around and that it is usually rather unpleasant to work in them, they can still be useful.

1. Classical propositional logic

A Hilbert-style proof calculus consists of:

1. A collection of *axiom schemes*. An axiom scheme is a logical scheme all whose instances are axioms.
2. A collection of *inference rules*. An inference rule is a schema that tells one how one can derive new formulas from formulas that have already been derived.

An example of a Hilbert-style proof system for classical propositional logic is the following. The axiom schemes are:

$$
\begin{array}{l}
\varphi \to (\psi \to \varphi) \\
(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \\
\varphi \to \varphi \vee \psi \\
\psi \to \varphi \vee \psi \\
(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi)) \\
\varphi \wedge \psi \to \varphi \\
\varphi \wedge \psi \to \psi \\
\varphi \to (\psi \to (\varphi \wedge \psi)) \\
\neg\neg\varphi \to \varphi
\end{array}
$$

The first axiom will be called **K** and the second **S**. The final axiom is **DNE**, for double negation elimination. The only inference rule will be Modus Ponens, which says that if one has already derived φ and $\varphi \to \psi$, then one is allowed to infer ψ. More formally, inductively define $\vdash \varphi$ as follows:

1. If φ is a substitution instance of an axiom scheme (i.e., an axiom), then $\vdash \varphi$.
2. If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.

More generally, we define $\Gamma \vdash \varphi$, as follows:

a. if $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$;
b. if φ is a substitution instance of an axiom scheme, then $\Gamma \vdash \varphi$;

c. if $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, then $\Gamma \vdash \psi$.

One can now show that $\Gamma \vdash \varphi$ is derivable in the Hilbert-style proof calculus if and only if it is derivable using classical natural deduction. To see this, we first need two lemmas:

Lemma 1.1. *If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \varphi$ is derivable in the Hilbert-style proof calculus, then so is $\Delta \vdash \varphi$.*

Proof. Easy proof by induction on the derivation of $\Gamma \vdash \varphi$. □

Lemma 1.2. *We have $\vdash \varphi \to \varphi$ for the Hilbert-style proof calculus.*

Proof. Note that both
$$\varphi \to \big((\varphi \to \varphi) \to \varphi\big)$$
and
$$\varphi \to (\varphi \to \varphi)$$
are instance of the **K**-axiom. As
$$\big(\varphi \to ((\varphi \to \varphi) \to \varphi)\big) \to \big((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)\big)$$
is an instance of the **S**-axiom, we obtain $\varphi \to \varphi$ by applying the Modus Ponens Rule twice. □

Theorem 1.3. (Deduction Theorem) *In the Hilbert-style proof calculus we have $\Gamma, \varphi \vdash \psi$ if and only if we have $\Gamma \vdash \varphi \to \psi$.*

Proof. We first prove the right-to-left direction: if $\Gamma \vdash \varphi \to \psi$, then also $\Gamma, \varphi \vdash \varphi \to \psi$ by the first lemma. Since also $\Gamma, \varphi \vdash \varphi$ by (a), we obtain $\Gamma, \varphi \vdash \psi$ by (c).

The left-to-right direction is proved by induction on the derivation of $\Gamma, \varphi \vdash \psi$. There are three cases:

1. $\psi \in \Gamma \cup \{\varphi\}$. This splits in two subcases: if $\psi \in \Gamma$, then we can use rule (a). If $\psi = \varphi$, then we can use the second lemma.
2. ψ is an axiom. Then $\Gamma \vdash \psi$ by (b). In addition, we have $\Gamma \vdash \psi \to (\varphi \to \psi)$, since $\psi \to (\varphi \to \psi)$ is an instance of the **K**-axiom, so we obtain $\Gamma \vdash \varphi \to \psi$ by Modus Ponens.
3. $\Gamma, \varphi \vdash \psi$ is obtained via the Modus Ponens rule, from $\Gamma, \varphi \vdash \chi \to \psi$ and $\Gamma, \varphi \vdash \chi$, say. In this case we can use the induction hypothesis to conclude $\Gamma \vdash \varphi \to (\chi \to \psi)$ and $\Gamma \vdash \varphi \to \chi$. Since $(\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$ is an instance of the **S**-axiom, we get $\Gamma \vdash \varphi \to \psi$ by two applications of the Modus Ponens Rule.

□

Theorem 1.4. *We can derive $\Gamma \vdash \varphi$ in the Hilbert-style calculus if and only if it is derivable in the natural deduction system for classical propositional logic.*

Proof. Suppose that $\Gamma \vdash \varphi$ is provable in the Hilbert-style calculus. By induction on the derivation of $\Gamma \vdash \varphi$ one shows that one can also derive $\Gamma \vdash \varphi$ using natural deduction, using that all axioms in the Hilbert-style calculus are derivable in classical natural deduction and using $\to$-elimination to take care of the Modus Ponens step.

Conversely, if $\Gamma \vdash \varphi$ is derivable in classical natural deduction, then one shows, again by induction on the derivation, that it is also derivable in the Hilbert-style proof calculus. This is fairly direct using the Deduction Theorem and the fact that the axioms other than **K** and **S** for our Hilbert calculus are basically the same as the rules for natural deduction. □

2. Intuitionistic propositional logic

One obtains a Hilbert-style proof calculus for intuitionistic propositional logic by replacing the axiom scheme for double negation elimination

$$\neg\neg\varphi \to \varphi$$

by that for ex falso:

$$\bot \to \varphi.$$

All the proofs that we gave in this chapter (Deduction Theorem and equivalence to natural deduction) work for this intuitionistic system as well.

Exercise 1. Give a derivation of $\bot \to \varphi$ in the Hilbert-style calculus for classical propositional logic.

CHAPTER 4

Negative translation

It is natural to think of classical logic as an extension of intuitionistic logic as it can be obtained from intuitionistic logic by adding an additional axiom (for instance, the Law of Excluded Middle $\varphi \vee \neg\varphi$). However, the opposite point of view makes sense as well: one could also think of intuitionistic logic as an extension of classical logic. The reason for this is that there is a faithful copy of classical logic inside intuitionistic logic: such a copy is called a *negative translation*. Below we will present a negative translation due to Gödel and Gentzen; but first, we will introduce the concept of a *nucleus*, which will be useful later.

1. Nuclei

We will work in the natural deduction system for intuitionistic propositional logic.

Definition 1.1. Let ∇ be a function mapping formulas in propositional logic to formulas in propositional logic. Such a mapping is called a *nucleus* if the following statements are provable in intuitionistic logic:

$$\begin{array}{l} \vdash_{\mathrm{IL}} \varphi \to \nabla\varphi \\ \vdash_{\mathrm{IL}} \nabla(\varphi \wedge \psi) \leftrightarrow (\, \nabla\varphi \wedge \nabla\psi \,) \\ \vdash_{\mathrm{IL}} (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi) \end{array}$$

Exercise 2. (a) Show that $\nabla\varphi{:}= \neg\neg\varphi$ is a nucleus. (You have already done this exercise, I hope.) This is the double negation nucleus and we will denote it by $\neg\neg$.

(b) Fix a propositional formula A. Show that also $\nabla\varphi{:}= \varphi \vee A$, $\nabla\varphi = A \to \varphi$, $\nabla\varphi{:}= (\varphi \to A) \to A$ are nuclei.

(c) Show that if ∇ is a nucleus, then $\vdash_{\mathrm{IL}} (\varphi \to \psi) \to (\nabla\varphi \to \nabla\psi)$ and $\vdash_{\mathrm{IL}} \nabla\varphi \leftrightarrow \nabla\nabla\varphi$ for all propositional formulas φ and ψ.

Given a nucleus ∇, let φ^∇ be the formula obtained from φ by applying ∇ to each propositional variable and each disjunction. More precisely, let φ^∇ be defined by induction on the structure of φ as follows:

$$\begin{array}{rcll} \varphi^\nabla & := & \nabla\varphi & \text{if } \varphi \text{ is a propositional variable or } \bot, \\ (\varphi \wedge \psi)^\nabla & := & \varphi^\nabla \wedge \psi^\nabla, & \\ (\varphi \vee \psi)^\nabla & := & \nabla(\varphi^\nabla \vee \psi^\nabla), & \\ (\varphi \to \psi)^\nabla & := & \varphi^\nabla \to \psi^\nabla. & \end{array}$$

Exercise 3. (a) Show that for any formula φ we have $\vdash_{\mathrm{IL}} \nabla\varphi^\nabla \leftrightarrow \varphi^\nabla$.

(b) Show that $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{IL}} \psi$ implies $\varphi_1^\nabla, \ldots, \varphi_n^\nabla \vdash_{\mathrm{IL}} \psi^\nabla$.

Exercise 4. In this exercise we will only consider nuclei of the form $\nabla\varphi{:}= (\varphi \to A) \to A$ for some fixed propositional formula A.

(a) Show $\vdash_{\mathrm{IL}} \nabla\bot \leftrightarrow A$.

(b) Show that $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{CL}} \psi$ implies $\varphi_1^\nabla, \ldots, \varphi_n^\nabla \vdash_{\mathrm{IL}} \psi^\nabla$.
(c) Show that if $A = \bot$, that is, if $\nabla\varphi = \neg\neg\varphi$, then $\vdash_{\mathrm{CL}} \varphi \leftrightarrow \varphi^\nabla$.

2. The Gödel-Gentzen negative translation

The previous exercise shows that $\varphi \mapsto \varphi^{\neg\neg}$ is a *negative translation*.

Definition 2.1. A mapping $(-)^*$ sending formulas in propositional logic to formulas in propositional logic is called a *negative translation* if the following hold for all Γ and φ:

(i) If $\Gamma \vdash_{\mathrm{CL}} \psi$, then $\Gamma^* \vdash_{\mathrm{IL}} \varphi^*$.
(ii) $\vdash_{\mathrm{CL}} \varphi \leftrightarrow \varphi^*$.

The mapping $\varphi \mapsto \varphi^{\neg\neg}$ is also called the *Gödel-Gentzen negative translation*.

Exercise 5. (a) Find intuitionistic natural deduction proofs of

$$\neg\neg(\varphi \to \psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$$

and

$$\neg(\neg\varphi \wedge \neg\psi) \leftrightarrow \neg\neg(\varphi \vee \psi).$$

(b) Show that $\vdash_{\mathrm{IL}} \varphi^{\neg\neg} \leftrightarrow \neg\neg\varphi$ and deduce that also $\varphi^* = \neg\neg\varphi$ is a negative translation.

Remark 2.2. Everything we said in this chapter will still be true when we turn to predicate logic, *except for part (b) of the previous exercise*. This means that a suitable extension of $\varphi^{\neg\neg}$ will still define a negative translation, but it will no longer be equivalent to $\neg\neg\varphi$. Indeed, there are examples of formulas φ that are tautologies in classical predicate logic, but for which $\neg\neg\varphi$ is not a tautology in intuitionistic predicate logic.