CHAPTER 2

Natural deduction

We introduce our first proof calculus: Gentzen's natural deduction.

1. Classical natural deduction

A natural deduction proof has the shape of a tree in which the nodes are decorated with formulas. However, as we are logicians we do not draw such trees in the following manner

$$\begin{tikzcd}
 & \varphi \arrow[dr] & & \psi \arrow[dl] & \gamma \arrow[d] & \chi \arrow[dl] & \alpha \arrow[dll] \\
\alpha \arrow[drr] & & \beta \arrow[d] & & \varphi \arrow[dll] & & \\
 & & \psi, & & & &
\end{tikzcd}$$

but instead like this:

$$\dfrac{\alpha \qquad \dfrac{\varphi \qquad \psi}{\beta} \qquad \dfrac{\gamma \qquad \chi \qquad \alpha}{\varphi}}{\psi.}$$

In a natural deduction proof the formula occurring at the root of the tree is called the *conclusion*, while the formulas at the leaves of the tree are its *assumptions*. In a natural deduction proof the assumptions can be of two kinds: *canceled* and *uncanceled*. When one starts building ones proof tree all assumptions are uncanceled, but in certain inferences one is allowed to cancel certain assumptions: the idea is that by making this inference something which was assumption is no longer one. The prime example is the rule which introduces an implication: suppose you are able to prove ψ assuming φ. Then you are allowed to deduce that $\varphi \to \psi$ holds, but that conclusion no longer depends on the assumption φ. To indicate that a formula has been canceled we will put square brackets around it, like in $[\varphi]$. Naturally, a natural deduction proof shows $\Gamma \vdash \varphi$ if its conclusion is φ and any *uncanceled* assumption belongs to Γ.

The class of all proof trees is defined inductively as follows:

0. Each formula φ is a proof tree, with uncancelled assumption and conclusion φ. (This is the only axiom.)

1a. If $\mathcal{D}_1$ is a proof tree with conclusion φ_1 and $\mathcal{D}_2$ is a proof tree with conclusion φ_2, then also

$$\dfrac{\begin{matrix}\mathcal{D}_1 \\ \varphi_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}$$

is a proof tree. (This rule is called $\wedge$-*introduction*.)

1b. If $\mathcal{D}$ is a proof tree with conclusion $\varphi \wedge \psi$, then also

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\varphi} \quad \text{and} \quad \frac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\psi}$$

are proof trees. (This rule is called $\wedge$-*elimination*.)

2a. If $\mathcal{D}$ is a proof tree with conclusion ψ, then also

$$\frac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi \to \psi}$$

is a proof tree; here by putting a $[\varphi]$ on top of $\mathcal{D}$ we mean that any occurence of the assumption φ in $\mathcal{D}$ may now be cancelled (see also the remark below). (This rule is called $\to$-*introduction*.)

2b. If $\mathcal{D}_1$ is a proof tree with conclusion φ and $\mathcal{D}_2$ is a proof tree with conclusion $\varphi \to \psi$, then also

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \varphi \to \psi\end{array}}{\psi}$$

is a proof tree. (This rule is called $\to$-*elimination*.)

3a. If $\mathcal{D}_1$ is a proof tree with conclusion φ and $\mathcal{D}_2$ is a proof tree with conclusion ψ then both

$$\frac{\begin{array}{c}\mathcal{D}_1\\ \varphi\end{array}}{\varphi \vee \psi} \quad \text{and} \quad \frac{\begin{array}{c}\mathcal{D}_2\\ \psi\end{array}}{\varphi \vee \psi}$$

are proof trees with conclusion $\varphi \vee \psi$. (This is rule is called $\vee$-*introduction*.)

3b. If $\mathcal{D}$ is a proof tree with conclusion $\varphi_1 \vee \varphi_2$ and both $\mathcal{D}_1$ and $\mathcal{D}_2$ are proof trees with conclusion χ, then also

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi_1 \vee \varphi_2\end{array} \qquad \begin{array}{c}[\varphi_1]\\ \mathcal{D}_1\\ \chi\end{array} \qquad \begin{array}{c}[\varphi_2]\\ \mathcal{D}_2\\ \chi\end{array}}{\chi}$$

is a proof tree, where one is allowed to cancel any occurrence of the assumption φ_i in $\mathcal{D}_i$ (see remark below). (This ruled is called $\vee$-*elimination*.)

4. If $\mathcal{D}$ is a proof tree with conclusion $\perp$ and φ is any formula, then also

$$\frac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \perp\end{array}}{\varphi}$$

is a proof tree in which one is is allowed to cancel any occurrence of the assumption $\neg\varphi$. (This is the *reductio ad absurdum rule*.)

REMARK 1.1. In any of the rules in which one may cancel any occurrence of some formula φ as an assumption in some part of the proof tree, we will work with the convention that one need not cancel all these occurrences. Some authors work with the opposite convention ("'the total discharge convention") in which one *has to* cancel all of them. It should be sort of obvious that this does not really matter.

What *does* matter, however, is the following: these rules also apply even when there are *no* occurrences of the formula φ as an uncanceled assumption. So, for example, if one has a proof tree $\mathcal{D}$ with conclusion ψ and there is no occurrence of the formula φ as an uncanceled assumption, one may still use the $\to$-introduction rule to derive $\varphi \to \psi$. (Exercise: give a natural deduction proof of $\psi \to (\varphi \to \psi)$.)

This also means that the following *ex falso rule* is a special case of the reduction ad absurdum rule: If $\mathcal{D}$ is a proof tree with conclusion $\perp$ and φ is any formula, then also

$$\dfrac{\begin{matrix}\mathcal{D}\\ \perp\end{matrix}}{\varphi}$$

is a proof tree.

THEOREM 1.2. (Soundness) *If there is a proof tree for* $\Gamma \vdash \varphi$ *in classical natural deduction, then* $\Gamma \models_{\mathrm{CL}} \varphi$.

PROOF. By induction on the construction of the proof tree. □

2. Intuitionistic natural deduction

Intuitionistic natural deduction is obtained by replacing the reductio ad absurdum rule by the weaker *ex falso rule*:

4. If $\mathcal{D}$ is a proof tree with conclusion $\perp$ and φ is any formula, then also

$$\dfrac{\begin{matrix}\mathcal{D}\\ \perp\end{matrix}}{\varphi}$$

is a proof tree. (This is the *ex falso rule*.)

THEOREM 2.1. (Soundness) *If there is a proof tree for* $\Gamma \vdash \varphi$ *in intuitionistic natural deduction, then* $\Gamma \models_{\mathrm{IL}} \varphi$.

PROOF. By induction on the construction of the proof tree. □