7th Exercise sheet Proof Theory
9 Dec 2014

Exercise 1 (a) Show that for any simple formula φ with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ there is a closed term $\mathbf{d}$ of type $\sigma \to (\sigma_2 \to \ldots \to (\sigma_n \to 0))$ in the language of HA^ω such that

$$\mathsf{HA}^\omega \vdash \forall x_1^{\sigma_1}, \ldots, x_n^{\sigma_n} \big(\varphi \leftrightarrow \mathbf{d} x_1 \ldots x_n =_0 0 \big).$$

Hint: Use induction on the structure of φ.

(b) Deduce that for any simple formula φ with free variables among $x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ we have

$$\mathsf{HA}^\omega \vdash \forall x_1^{\sigma_1}, \ldots, x_n^{\sigma_n} \big(\varphi \vee \neg\varphi \big).$$

Exercise 2 Let ∇ be a function mapping formulas in the language of HA^ω to formulas in the language in HA^ω, such that the following statements are provable in HA^ω:

$$\begin{aligned}
&\mathsf{HA}^\omega \vdash \varphi \to \nabla\varphi \\
&\mathsf{HA}^\omega \vdash \nabla(\varphi \wedge \psi) \leftrightarrow (\, \nabla\varphi \wedge \nabla\psi \,) \\
&\mathsf{HA}^\omega \vdash (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi)
\end{aligned}$$

In addition, let φ^∇ be the formula obtained from φ by applying ∇ to each atomic subformula, disjunction and existentially quantified subformula. More precisely, φ^∇ is defined by induction on the structure of φ as follows:

$$\begin{aligned}
\varphi^\nabla &:= \nabla\varphi \qquad \text{if } \varphi \text{ is an atomic formula,} \\
(\varphi \wedge \psi)^\nabla &:= \varphi^\nabla \wedge \psi^\nabla, \\
(\varphi \vee \psi)^\nabla &:= \nabla(\varphi^\nabla \vee \psi^\nabla), \\
(\varphi \to \psi)^\nabla &:= \varphi^\nabla \to \psi^\nabla, \\
(\forall x^\sigma\, \varphi(x)\,)^\nabla &:= \forall x^\sigma\, (\varphi(x))^\nabla, \\
(\exists x^\sigma\, \varphi(x)\,)^\nabla &:= \nabla\exists x^\sigma.\, (\varphi(x))^\nabla.
\end{aligned}$$

Show that $\mathsf{HA}^\omega \vdash \varphi$ implies $\mathsf{HA}^\omega \vdash \varphi^\nabla$.

Exercise 3 (Tricky!) According to Gödel's Incompleteness Theorem, there is a simple formula $\varphi(x^0)$ with only x^0 free such that $\mathsf{PA}^\omega \vdash \varphi(t)$ for all closed terms t of type 0, while at the same time $\mathsf{PA}^\omega \nvdash \forall x^0\, \varphi(x)$ (the idea being that $\varphi(x)$ says that x is not the code of a proof of the inconsistency of PA^ω). Deduce from this that there is a formula $\psi(x^0)$ such that $\mathsf{PA}^\omega \vdash \exists x^0\, \psi(x^0)$, while at the same time there is no closed term t of type 0 such that $\mathsf{PA}^\omega \vdash \psi(t)$.