5th Exercise sheet Proof Theory
25 Nov 2015

Exercise 1 Consider the following De Morgan laws:

$$\neg\exists x\,\varphi \to \forall x\,\neg\varphi$$
$$\forall x\,\neg\varphi \to \neg\exists x\,\varphi$$
$$\neg\forall x\,\varphi \to \exists x\,\neg\varphi$$
$$\exists x\,\neg\varphi \to \neg\forall x\,\varphi$$

Which ones of those are also intuitionistically valid? Give an intuitionistic natural deduction-style proof, if possible; if this is not possible, give a classical proof and a Kripke model refuting the statement.

Exercise 2 Consider the following classical tautologies:

$$(\forall x\,\varphi \to \psi) \to \exists x\,(\varphi \to \psi)$$
$$(\exists x\,\varphi \to \psi) \to \forall x\,(\varphi \to \psi)$$
$$(\psi \to \forall x\,\varphi) \to \forall x(\psi \to \varphi)$$
$$(\psi \to \exists x\,\varphi) \to \exists x(\psi \to \varphi)$$

(Here ψ is a formula, while φ is a semi-formula with $F(\varphi) \cap \mathrm{Var} = \{x\}$.) Which ones of those are also intuitionistically valid? Give an intuitionistic natural deduction-style proof, if possible; if this is not possible, give a classical proof and a Kripke model refuting the statement.

Exercise 3 Give an (effective!) proof of the existence property: if a sequent $\Gamma \Rightarrow \exists x\,\varphi$ is derivable in the intuitionistic sequent calculus *à la* Gentzen and Γ does not contain any existential quantifiers and disjunctions, then there is a term t such that $\Gamma \Rightarrow \varphi(t)$ is derivable.

Exercise 4 We extend the theory of nuclei to predicate logic. So now a nucleus is a function ∇ sending semi-formulas in predicate logic to semi-formulas in predicate logic, in such a way that the following statements are provable in intuitionistic logic:

$$\vdash_{\mathrm{IL}} \varphi \to \nabla\varphi$$
$$\vdash_{\mathrm{IL}} \nabla(\varphi \wedge \psi) \leftrightarrow (\,\nabla\varphi \wedge \nabla\psi\,)$$
$$\vdash_{\mathrm{IL}} (\varphi \to \nabla\psi) \to (\nabla\varphi \to \nabla\psi)$$

In addition, define φ^∇ by induction on the structure of φ as follows:

$$
\begin{array}{rcll}
\varphi^\nabla & := & \nabla\varphi & \text{if } \varphi \text{ is a propositional variable or } \bot, \\
(\varphi \wedge \psi)^\nabla & := & \varphi^\nabla \wedge \psi^\nabla, & \\
(\varphi \vee \psi)^\nabla & := & \nabla(\varphi^\nabla \vee \psi^\nabla), & \\
(\varphi \to \psi)^\nabla & := & \varphi^\nabla \to \psi^\nabla, & \\
(\forall x\, \varphi(x)\,)^\nabla & := & \forall x\, (\varphi(x))^\nabla, & \\
(\exists x\, \varphi(x)\,)^\nabla & := & \nabla\exists x\, (\varphi(x))^\nabla. &
\end{array}
$$

(a) Show that for any formula φ we have $\vdash_{\mathrm{IL}} \nabla\varphi^\nabla \leftrightarrow \varphi^\nabla$.

(b) Show that $\varphi_1, \ldots, \varphi_n \vdash_{\mathrm{IL}} \psi$ implies $\varphi_1^\nabla, \ldots, \varphi_n^\nabla \vdash_{\mathrm{IL}} \psi^\nabla$.