2nd Exercise sheet Proof Theory
2 Nov 2015

Exercise 1 Give natural deduction proofs of the following statements, using only those rules that are intuitionistically valid:

(a) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$.

(b) $\varphi \to \neg\neg\varphi$.

(c) $\neg\neg\neg\varphi \to \neg\varphi$.

(d) $(\varphi \to \neg\neg\psi) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$.

(e) $\neg\neg(\varphi \wedge \psi) \leftrightarrow \neg\neg\varphi \wedge \neg\neg\psi$.

Exercise 2 Consider the following De Morgan laws:

$$\begin{array}{l} \neg(\varphi \vee \psi) \to \neg\varphi \wedge \neg\psi \\ \neg\varphi \wedge \neg\psi \to \neg(\varphi \vee \psi) \\ \neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi \\ \neg\varphi \vee \neg\psi \to \neg(\varphi \wedge \psi) \end{array}$$

Give natural deduction proofs of these laws, using the Reductio ad Absurdum rule instead of the Ex Falso rule only when this is unavoidable.

Exercise 3 Give proofs of the following formulas in classical natural deduction.

(a) $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$.

(b) $((\varphi \to \psi) \to \varphi) \to \varphi$.

Exercise 4 (a) Give natural deduction-style proofs in intuitionistic logic of

$$\neg\neg(\varphi \vee \neg\varphi)$$
$$(\varphi \vee \neg\varphi) \to (\neg\neg\varphi \to \varphi)$$

(b) Suppose that in the natural deduction system for classical logic we would replace the reductio ad absurdum rule with a rule saying that for any φ the statement $\varphi \vee \neg\varphi$ is an axiom (so for any axiom φ we have a proof tree

$$\frac{}{\varphi \vee \neg\varphi}$$

with conclusion $\varphi \vee \neg\varphi$ and no uncanceled assumptions). Deduce from (a) that this new system for natural deduction proves the same statements $\Gamma \vdash \varphi$ as the old one.

(c) Give a Kripke model refuting the intuitionistic validity of

$$(\neg\neg p \to p) \to (p \vee \neg p),$$

thus showing that the law of excluded middle and the law of double negation elimination $\neg\neg\varphi \to \varphi$ are not "instancewise" equivalent.