1st Exercise sheet Proof Theory
28 Oct 2015

Exercise 1 Consider the following De Morgan laws:

$$\neg(\varphi \vee \psi) \to \neg\varphi \wedge \neg\psi$$
$$\neg\varphi \wedge \neg\psi \to \neg(\varphi \vee \psi)$$
$$\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$$
$$\neg\varphi \vee \neg\psi \to \neg(\varphi \wedge \psi)$$

Which ones of these are also intuitionistic tautologies? Justify your answer using Kripke models.

Exercise 2 Another possible ("global") definition of $\Gamma \models_{\mathrm{IL}} \varphi$ could have been: if one has a Kripke model and all the formulas from Γ are forced in all worlds in this model, also φ is forced in all worlds. Show that this coincides with the "local" definition given in the lecture (i.e.: if one has a world w in a Kripke model and all Γ are forced in w, then also φ is forced in w).

Exercise 3 Let (W, R, f) be a Kripke model and let $\sim$ be the relation on W defined by: $x \sim y$ if $R(x, y)$ and $R(y, x)$. Check that $\sim$ is an equivalence relation and write $[w]$ for the $\sim$-equivalence class of w. Show that that there is a natural Kripke model $(W/\sim, R', f')$ with set of worlds $W/\sim$, such that $[w] \Vdash \varphi$ in $(W/\sim, R', f')$ if and only if $w \Vdash \varphi$ in (W, R, f). Conclude that intuitionistic propositional logic is also complete with respect to Kripke models (W, R, f) in which R is not only reflexive and transitive, but also anti-symmetric.

Exercise 4 Give a semantic proof of the fact that intuitionistic propositional logic has the disjunction property: if $\varphi \vee \psi$ is an intuitionistic tautology, then so is at least one of φ and ψ. Why does this fail for classical logic?