6th Homework sheet Proof Theory

- Deadline: 12 December.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture*.
- Good luck!

(a) *(40 points)* Show that there is a closed term **lt** of type $0 \to (0 \to 0)$ such that

$$\mathsf{HA}^\omega \vdash \forall x^0, y^0 \left(\mathbf{lt}\, x\, y = 0 \leftrightarrow \exists n^0\, \mathbf{plus}\, x\, Sn = y \right)$$

Instead of $\mathbf{lt}\, x\, y = 0$ we will write $x < y$ and we will write $x \leq y$ as abbreviation for $x = y \vee x < y$.

(b) *(30 points)* Consider the following statement:

$$(\clubsuit) \qquad \forall f^{0\to 0}\, \exists x_1^0, x_2^0, x_3^0 \left(x_1 < x_2 < x_3 \wedge f(x_1) \leq f(x_2) \leq f(x_3) \right).$$

Give a classical proof, and argue as follows. Let f be an arbitrary object of type $0 \to 0$. From the Law of Excluded Middle and the De Morgan laws we get:

$$(\exists x^0\, \forall y^0\, f(y) \leq x) \vee (\forall x^0\, \exists y^0\, x < f(y)).$$

Then show $(\clubsuit)$ in both cases.

(c) *(30 points)* Show that $(\clubsuit)$ is also provable constructively. How does your constructive argument find suitable x_1, x_2, x_3 from f?