Slides for a course on model theory

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Goal

Recall our goal was to prove:

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension.

We first prove a lemma.

A lemma

Lemma

Let A be an L-structure. There exists an elementary extension B of A such that for every subset $X \subseteq A$, every 1-type in L_X which is finitely satisfied in $(A, a)_{a \in X}$ is realized in $(B, a)_{a \in X}$.

Proof.

Let $(\Gamma_i(x_i))_{i \in I}$ be the collection of all such 1-types and b_i be new constants. Then every finite subset of

$$\Gamma := \bigcup_{i \in I} \Gamma_i(b_i)$$

is satisfied in $(A, a)_{a \in A}$, so it has a model B. Since Γ contains $\mathrm{ElDiag}(A)$, the model A embeds into B. $\square$

Existence of rich models

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension.

Proof.

Let A be an L-structure. We will build an elementary chain of L-structures $(A_i : i \in \kappa^+)$. We set $A_0 = A$, at successor stages we apply the previous lemma and at limit stages we take the colimit. Now let B be the colimit of the entire chain. We claim B is κ^+-saturated (which is more than we need).

So let $X \subseteq B$ be a subset of cardinality $< \kappa^+$ and $\Gamma(x)$ be a 1-type in L_X that is finitely satisfied in $(A, a)_{a \in X}$. Since κ^+ is regular, there is an $i \in \kappa^+$ such that $X \subseteq A_i$. And since A embeds elementarily into A_i, the type $\Gamma(x)$ is also finitely satisfied in $(A_i, a)_{a \in X}$. So it is realized in A_{i+1}, and therefore also in B, because A_{i+1} embeds elementarily into B. $\square$

Even richer models

Now that we have this we can be even more ambitious:

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension all whose reducts are strongly κ-homogeneous.

We need a lemma:

Lemma

Suppose A is κ-saturated and B is an elementary substructure of A satisfying $|B| < \kappa$. Then any elementary map f between subsets of B can be extended to an elementary embedding of B into A.

Proof.

If $f : S \to B$ is the elementary mapping, then $(B, b)_{b \in S} \equiv (A, f(b))_{b \in S}$. Since $|S| < \kappa$, also $(A, f(b))_{b \in S}$ is κ-saturated und hence κ^+-universal. So $(B, b)_{b \in S}$ embeds elementarily into $(A, f(b))_{b \in S}$: so we have an elementary embedding of B into A extending f. $\square$

Existence of very rich models

Theorem

For every infinite cardinal number κ, every structure has a κ-saturated elementary extension all whose reducts are strongly κ-homogeneous.

Proof.

Let A be an L-structure. Again, we will build an elementary chain of L-structures $(M_\alpha : \alpha \in \kappa^+)$. We set $M_0 = A$, at successor stages $\alpha + 1$ we take an $|M_\alpha|^+$-saturated elementary extension of M_α and at limit stages we take the colimit. Now let M be the colimit of the entire chain. We claim M is as desired.

Any subset of S of M that has cardinality $\leq \kappa$, must be a subset of some M_α (using again that κ^+ is regular). So M is κ^+-saturated. It remains to show that every reduct of M is strongly κ-homogeneous. $\square$

Existence of very rich models, proof finished

Proof.

Let f be any mapping between subsets of M that is elementary, with domain and range having cardinality $< \kappa$. Again, domain and range will belong to some M_α. Without loss of generality we may assume that α is a limit ordinal. We extend f to a map $f_\alpha : M_\alpha \to M_{\alpha+1}$ using the lemma.

We will build maps f_β for all $\alpha \leq \beta < \kappa^+$ in such a way that f_β is an elementary embedding of M_β in $M_{\beta+1}$ and $f_{\beta+1}$ extends f_β^{-1}. It follows that $f_{\beta+2}$ extends f_β and that the union h over all f_β with β even is an automorphism of M.

The construction is: At limit stages we take unions over all previous even stages. And at successor stages we apply the lemma.

This argument works equally well for reducts of M. □

Definability

Definition

Let A be an L-structure and $R \subseteq A^n$ be a relation. The relation R is called *definable*, if there a formula $\varphi(x_1, \ldots, x_n)$ such that

$$R = \{(a_1, \ldots, a_n) \in A^n : A \models \varphi(a_1, \ldots, a_n)\}.$$

A homomorphism $f : A \to A$ *leaves* R *setwise invariant* if $\{(f(a_1), \ldots, f(a_n) : (a_1, \ldots, a_n) \in R\} = R$.

Proposition

Every elementary embedding from A to itself leaves all definable relations setwise invariant.

Definability results

Theorem

Let L be a language and P a predicate not in L. Suppose (A, R) is an ω-saturated $L \cup \{P\}$-structure and that A is strongly ω-homogeneous. Then the following are equivalent:

(1) R is definable in A.

(2) every automorphism of A leaves R setwise invariant.

Proof.

(1) $\Rightarrow$ (2) always holds, because automorphisms are elementary embeddings.

(2) $\Rightarrow$ (1): Suppose R is not definable. By the next lemma there are tuples a and b having the same type such that $R(a)$ is true and $R(b)$ is false. But then there is an automorphism of A that sends a to b by strong homogeneity. So R is not setwise invariant under automorphisms of A. □

A lemma

Lemma

Suppose A is a structure and R is not definable in A. If (A, R) is ω-saturated, then there are tuples a and b having the same n-type in A such that $R(a)$ is true and $R(b)$ is false.

Proof.

First consider the type $\Sigma(x) = \{\varphi(x) \in L : (A, R) \models \forall x \left(\neg P(x) \to \varphi(x)\} \cup \{P(x)\}$. This type is finitely satisfiable in (A, R): for if not, then there would be a formula $\varphi(x)$ such that $(A, R) \models \neg P(x) \to \varphi(x)$ and $(A, R) \models \neg(\varphi(x) \wedge P(x))$. But then $\neg\varphi(x)$ would define R. By ω-saturation, there is an element a realizing $\Sigma(x)$. Now consider the type $\Gamma(x) = \mathrm{tp}_A(a) \cup \{\neg P(x)\}$. This type is also finitely satisfiable in (A, R): for if not, then there would be a formula $\varphi(x) \in L$ such that $(A, R) \models \varphi(a)$ and $(A, R) \models \neg(\varphi(x) \wedge \neg P(x))$. This is impossible by construction of a. By ω-saturation there is an element b realizing $\Gamma(x)$. So we have that a and b have the same type in A, while $R(a)$ is true and $R(b)$ is false. $\square$

Svenonius' Theorem

Svenonius' Theorem

Let A be an L-structure and R be a relation on A. Then the following are equivalent:

(1) R is definable in A.

(2) every automorphism of an elementary extension (B, S) of (A, R) leaves S setwise invariant.

Proof.

(1) $\Rightarrow$ (2): If R is definable in A, then S is definable in B by the same formula; so it will be left setwise invariant by any automorphism.

(2) $\Rightarrow$ (1): Let (B, S) be an ω-saturated and strongly ω-homogeneous extension of (A, R). S will be definable in (B, S) by the previous theorem; but then R in A will be definable by the same formula. $\square$

Omitting types theorem

Definition

Let T be an L-theory and $\Sigma(x)$ be a partial type. Then $\Sigma(x)$ is *isolated in T* if there is a formula $\varphi(x)$ such that $\exists x\, \varphi(x)$ is consistent with T and

$$T \models \varphi(x) \to \sigma(x)$$

for all $\sigma(x) \in \Sigma(x)$.

Exercise

A type is isolated iff it is an isolated point in the type space $S_1(T)$.

Omitting types theorem

Let T be a consistent theory in a countable language. If a partial type $\Sigma(x)$ is not isolated in T, then there is a countable model of T which omits $\Sigma(x)$.

Reminder

Recall from *Grondslagen van de Wiskunde*:

Theorem

Suppose T is a consistent theory in a language L and C is a set of constants in L. If for any formula $\psi(x)$ in the language L there is a constant $c \in C$ such that

$$T \models \exists x\, \psi(x) \rightarrow \psi(c),$$

then T has a model whose universe consists entirely of interpretations of elements of C.

Proof.

Extend T to a maximally consistent theory and then build a model from the constants in C. $\square$

Omitting types theorem, proof

Omitting types theorem

Let T be a consistent theory in a countable language. If a partial type $\Sigma(x)$ is not isolated in T, then there is a countable model of T which omits $\Sigma(x)$.

Proof.

Let $C = \{c_i \,;\, i \in \mathbb{N}\}$ be a countable collection of fresh constants and L_C be the language L extending with these constants. Let $\{\psi_i(x) \,:\, i \in \mathbb{N}\}$ be an enumeration of the formulas with one free variable in the language L_C. We will now inductively create a sequence of sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$. The idea is to apply to previous theorem to $T \cup \{\varphi_0, \varphi_1, \ldots\}$.

If $n = 2i$, we take a fresh constant $c \in C$ (one that does not occur in φ_m with $m < n$) and put

$$\varphi_n = \exists x \psi_i(x) \to \psi(c).$$

This makes sure we can create a model from the constants in C. □

Omitting types theorem, proof finished

Proof.

If $n = 2i + 1$ we make sure that c_i omits $\Sigma(x)$, as follows. Consider $\delta = \bigwedge_{m<n} \varphi_m$. δ is really of the form $\delta(c_i, \overline{c})$ where $\overline{c}$ is a sequence of constants not containing c_i. Since $\Sigma(x)$ is not isolated, there must be a formula $\sigma(x) \in \Sigma(x)$ such that $T \not\models \exists\overline{y}\, \delta(x, \overline{y}) \rightarrow \sigma(x)$; in other words, such that $T \cup \{\exists\overline{y}\, \delta(x, y)\} \cup \{\neg\sigma(x)\}$ is consistent. Put $\varphi_n = \neg\sigma(c_i)$.

The proof is now finished by showing by induction that each $T \cup \{\varphi_0, \ldots, \varphi_n\}$ is consistent and then applying the theorem from *Grondslagen*. □

Exercises

Exercise

Prove the generalised omitting types theorem: Let T be a consistent theory in a countable language and let $\{\Gamma_i : i \in \mathbb{N}\}$ be a sequence of partial n_i-types (for varying n_i). If none of the Γ_i is isolated in T, then there is a countable model which omits all Γ_i.

Exercise

Let T be a complete theory. Show that models of T realise all isolated partial types.

Exercise

Prove that the omitting types theorem is specific to the countable case: give an example of a consistent theory T in an uncountable language and a partial type in T which is not isolated, but which is nevertheless realised in every model of T.