Types

Fix $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be a fixed sequence of distinct variables.

Definition

- A *partial n-type in L* is a collection of formulas $\varphi(x_1, \ldots, x_n)$ in L.
- If A is an L-structure and $a_1, \ldots, a_n \in A$, then the *type of* $(a_1, \ldots, a_n)$ *in* A is the set of L-formulas

$$\{\varphi(x_1, \ldots, x_n) : A \models \varphi(a_1, \ldots, a_n)\};$$

  we denote this set by $\mathrm{tp}_A(a_1, \ldots, a_n)$ or simply by $\mathrm{tp}(a_1, \ldots, a_n)$ if A is understood.
- A *n-type in L* is a set of formulas of the form $\mathrm{tp}_A(a_1, \ldots, a_n)$ for some L-structure A and some $a_1, \ldots, a_n \in A$.

Realizing and omitting types

Definition

- If $\Gamma(x_1, \ldots, x_n)$ is a partial n-type in L, we say $(a_1, \ldots, a_n)$ *realizes* Γ *in* A if every formula in Γ is true of $a_1, \ldots, a_n$ in A.
- If $\Gamma(x_1, \ldots, x_n)$ is a partial n-type in L and A is an L-structure, we say that Γ is *realized or satisfied in* A if there is some n-tuple in A that realizes Γ in A. If no such n-tuple exists, then we say that A *omits* Γ.
- If $\Gamma(x_1, \ldots, x_n)$ is a partial n-type in L and A is an L-structure, we say that Γ is *finitely satisfiable in* A if any finite subset of Γ is realized in A.

Exercises

Exercise

Show that a partial n-type is an n-type iff it is finitely satisfiable and contains $\varphi(x_1, \ldots, x_n)$ or $\neg\varphi(x_1, \ldots, x_n)$ for every L-formula φ whose free variables are among the fixed variables $x_1, \ldots, x_n$.

Exercise

Show that a partial n-type can be extended to an n-type iff it is satisfiable.

Exercise

Suppose $A \equiv B$. If $\Gamma(x_1, \ldots, x_n)$ is finitely satisfiable in A, then it is also finitely satisfiable in B.

Logic topology

Definition

Let T be a theory in L and let $\Gamma = \Gamma(x_1, \ldots, x_n)$ be a partial n-type in L.

- Γ *is consistent with* T if $T \cup \Gamma$ has a model.
- The set of all n-types consistent with T is denoted by $S_n(T)$. These are exactly the n-types in L that contain T.

The set $S_n(T)$ can be given the structure of a topological space, where the basic open sets are given by

$$[\varphi(x_1, \ldots, x_n)] = \{\Gamma(x_1, \ldots, x_n) \in S_n(T) : \varphi \in \Gamma\}.$$

This is called the *logic topology*.

Type spaces

Theorem

The space $S_n(T)$ with the logic topology is a totally disconnected, compact Hausdorff space. Its closed sets are the sets of the form

$$\{\Gamma \in S_n(T) : \Gamma' \subseteq \Gamma\}$$

where Γ' is a partial n-type. In fact, two partial n-types are equivalent over T iff they determine the same closed set. Furthermore, the clopen sets in the type space are precisely the ones of the form $[\varphi(x_1, \ldots, x_n)]$.

κ-saturated models

Let A be an L-structure and X a subset of A. We write L_X for the language L extended with constants for all elements of X and $(A, a)_{a \in X}$ for the L_X-expansion of A where we interpret the constant $a \in X$ as itself.

Definition

Let A be an L-structure and let κ be an infinite cardinal. We say that A is κ-*saturated* if the following condition holds: if X is any subset of A having cardinality $< \kappa$ and $\Gamma(x)$ is any 1-type in L_X that is finitely satisfiable in $(A, a)_{a \in X}$, then $\Gamma(x)$ is itself satisfied in $(A, a)_{a \in X}$.

Remark

1. If A is infinite and κ-saturated, then A has cardinality at least κ.
2. If A is finite, then A is κ-saturated for every κ.
3. If A is κ-saturated and X is a subset of A having cardinality $< \kappa$, then $(A, a)_{a \in X}$ is also κ-saturated.

Property of κ-saturated models

Theorem

Suppose κ is an infinite cardinal, A is κ-saturated and $X \subseteq A$ is a subset of cardinality $< \kappa$. Suppose $\Gamma(y_i : i \in I)$ is a collection of L_X-formulas with $|I| \leq \kappa$. If Γ is finitely satisfiable in $(A, a)_{a \in X}$, then Γ is satisfiable in $(A, a)_{a \in X}$.

Proof.

Without loss of generality we may assume that $I = \kappa$ and Γ is complete: contains either φ or $\neg\varphi$ for every L_X-formula φ with free variables among $\{y_i : i \in \kappa\}$.

Write $\Gamma_{\leq j}$ for the collection of those elements of Γ that only contain variables y_i with $i \leq j$. By induction on j we will find an element a_j such that $(a_i)_{i \leq j}$ realizes $\Gamma_{\leq j}$. Consider Γ' which is $\Gamma_{\leq j}$ with all y_i replaced by a_i for $i < j$. This is a 1-type which is finitely satisfiable in $(A, a)_{a \in X \cup \{a_i : i < j\}}$ (check!). Since $(A, a)_{a \in X \cup \{a_i : i < j\}}$ is κ-saturated, we find a suitable a_j. □