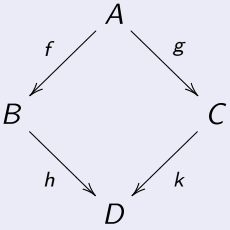
Amalgamation Theorem

Amalgamation Theorem

Let L_1, L_2 be languages and $L = L_1 \cap L_2$, and suppose A, B and C are structures in the languages L, L_1 and L_2, respectively. Any pair of L-elementary embeddings $f : A \to B$ and $g : A \to C$ fit into a commuting square



where D is an $L_1 \cup L_2$-structure, h is an L_1-elementary embedding and k is an L_2-elementary embedding.

Proof.

Immediate consequence of Robinson's Consistency Theorem. (Why?) □

Craig Interpolation

Craig Interpolation Theorem

Let φ and ψ be sentences in some language such that $\varphi \models \psi$. Then there is a sentence θ such that

1. $\varphi \models \theta$ and $\theta \models \psi$;
2. every predicate, function or constant symbol that occurs in θ occurs also in both φ and ψ.

Proof.

Let L be the common language of φ and ψ. We will show that $T_0 \models \psi$ where $T_0 = \{\sigma \in L : \varphi \models \sigma\}$. This is sufficient: for then there are $\theta_1, \ldots, \theta_n \in T_0$ such that $\theta_1, \ldots, \theta_n \models \psi$ by Compactness. So $\theta := \theta_1 \wedge \ldots \wedge \theta_n$ is the interpolant. □

Craig Interpolation, continued

Lemma

Let L be the common language of φ and ψ. If $\varphi \models \psi$, then $T_0 \models \psi$ where $T_0 = \{\sigma \in L : \varphi \models \sigma\}$.

Proof.

Suppose not. Then $T_0 \cup \{\neg\psi\}$ has a model A. Write $T = \mathrm{Th}_L(A)$. We now have $T_0 \subseteq T$ and:

1. T is a complete L-theory.
2. $T \cup \{\neg\psi\}$ is consistent (because A is a model).
3. $T \cup \{\varphi\}$ is consistent.

(Proof of 3: Suppose not. Then, by Compactness, there would a sentence $\sigma \in T$ such that $\varphi \models \neg\sigma$. But then $\neg\sigma \in T_0 \subseteq T$. Contradiction!)

Now we can apply Robinson's Consistency Theorem to deduce that $T \cup \{\neg\psi, \varphi\}$ is consistent. But that contradicts $\varphi \models \psi$. $\square$

Beth Definability Theorem

Definition

Let L be a language a P be a predicate symbol not in L, and let T be an $L \cup \{P\}$-theory. *T defines P implicitly* if any L-structure M has at most one expansion to an $L \cup \{P\}$-structure which models T. There is another way of saying this: let T' be the theory T with all occurrences of P replaced by P'. Then *T defines P implicitly* iff

$$T \cup T' \models \forall x_1, \ldots x_n \left(P(x_1, \ldots, x_n) \leftrightarrow P'(x_1, \ldots, x_n) \right).$$

T defines P explicitly, if there is an L-formula $\varphi(x_1, \ldots, x_n)$ such that

$$T \models \forall x_1, \ldots, x_n \left(P(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n) \right).$$

Beth Definability Theorem

T defines P implicitly if and only if T defines P explicitly.

(Right-to-left direction is obvious.)

Beth Definability Theorem, proof

Proof. Suppose T defines P implicitly. Add new constants $c_1, \ldots, c_n$ to the language. Then we have $T \cup T' \models P(c_1, \ldots, c_n) \to P'(c_1, \ldots, c_n)$. By Compactness and taking conjunctions we can find an $L \cup \{P\}$-formula ψ such that $T \models \psi$ and

$$\psi \wedge \psi' \models P(c_1, \ldots, c_n) \to P'(c_1, \ldots, c_n)$$

(where ψ' is ψ with all occurrences of P replaced by P'). Taking all the Ps to one side and the P's to another, we get

$$\psi \wedge P(c_1, \ldots, c_n) \models \psi' \to P'(c_1, \ldots, c_n)$$

So there is a Craig Interpolant θ such that

$$\psi \wedge P(c_1, \ldots, c_n) \models \theta \text{ and } \theta \models \psi' \wedge P'(c_1, \ldots, c_n)$$

By symmetry also

$$\psi' \wedge P'(c_1, \ldots, c_n) \models \theta \text{ and } \theta \models \psi \wedge P(c_1, \ldots, c_n)$$

So $\theta = \theta(c_1, \ldots, c_n)$ is, modulo T, equivalent to $P(c_1, \ldots, c_n)$ and $\theta(x_1, \ldots, x_n)$ defines P explicitly. □

Chang-Łoś-Suszko Theorem

Definition

A Π_2-*sentence* is a sentence which consists first of a sequence of universal quantifiers, then a sequence of existential quantifers and then a quantifier-free formula.

Definition

A theory T is *preserved by directed unions* if, for any directed system consisting of models of T and embeddings between them, also the colimit is a model T.

Chang-Łoś-Suszko Theorem

A theory is preserved under directed unions if and only if T can be axiomatised by Π_2-sentences.

Proof.

The easy direction is: Π_2-sentences are preserved by directed unions. We do the other direction. □

Chang-Łoś-Suszko Theorem, proof

Proof. Suppose T is preserved by direction unions. Again, let

$$T_0 = \{\varphi : \varphi \text{ is } \Pi_2 \text{ and } T \models \varphi\},$$

and let B be a model of T_0. We will construct a directed chain of embeddings

$$B = B_0 \to A_0 \to B_1 \to A_1 \to B_2 \to A_2 \dots$$

such that:

1. Each A_n is a model of T.
2. The composed embeddings $B_n \to B_{n+1}$ are elementary.
3. Every universal sentence in the language L_{B_n} true in B_n is also true in A_n (when regarding A_n is an L_{B_n}-structure via the embedding $B_n \to A_n$).

This will suffice, because when we take the colimit of the chain, then it is:

- the colimit of the A_n, and hence a model of T, by assumption on T.
- the colimit of the B_n, and hence elementary equivalent to each B_n.

So B is a model of T, as desired.

Chang-Łoś-Suszko Theorem, proof continued

Construction of A_n: We need A_n to be a model of T and every universal sentence in the language L_{B_n} true in B_n to be true in A_n as well. So let

$$T' = T \cup \{\varphi \in L_{B_n} : \varphi \text{ universal and } B_n \models \varphi\};$$

to show that T' is consistent. Suppose not. Then there is a universal sentence $\forall x_1, \ldots x_n \, \varphi(x_1, \ldots, x_n, b_1, \ldots, b_k)$ with $b_i \in B_n$ that is inconsistent with T. So

$$T \models \exists x_1, \ldots, x_n \neg\varphi(x_1, \ldots, x_n, b_1, \ldots, b_k)$$

and

$$T \models \forall y_1, \ldots, y_k \, \exists x_1, \ldots, x_n \neg\varphi(x_1, \ldots, x_n, y_1, \ldots, y_k)$$

because the b_i do not occur in T. But this contradicts the fact that B_n is a model of T_0.

Chang-Łoś-Suszko Theorem, proof finished

Construction of B_{n+1}: We need $A_n \to B_{n+1}$ to be an embedding and $B_n \to B_{n+1}$ to be elementary. So let

$$T' = \mathrm{Diag}(A_n) \cup \mathrm{ElDiag}(B_n)$$

(identifying the element of B_n with their image along the embedding $B_n \to A_n$); to show that T' is consistent. Suppose not. Then there is a quantifier-free sentence

$$\varphi(b_1, \ldots, b_n, a_1, \ldots, a_k)$$

with $b_i \in B_n$ and $a_i \in A_n \setminus B_n$ which is true in A_n, but is inconsistent with $\mathrm{ElDiag}(B_n)$. Since the a_i do not occur in B_n, we must have

$$B_n \models \forall x_1, \ldots, x_k \neg\varphi(b_1, \ldots, b_n, x_1, \ldots, x_k).$$

This contradicts the fact that all universal L_{B_n}-sentences true in B_n are also true in A_n. □

Types

Fix $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be a fixed sequence of distinct variables.

Definition

- A *partial n-type in L* is a collection of formulas $\varphi(x_1, \ldots, x_n)$ in L.
- If A is an L-structure and $a_1, \ldots, a_n \in A$, then the *type of* $(a_1, \ldots, a_n)$ *in* A is the set of L-formulas

  $$\{\varphi(x_1, \ldots, x_n) : A \models \varphi(a_1, \ldots, a_n)\};$$

  we denote this set by $\mathrm{tp}_A(a_1, \ldots, a_n)$ or simply by $\mathrm{tp}(a_1, \ldots, a_n)$ if A is understood.
- A *n-type in L* is a set of formulas of the form $\mathrm{tp}_A(a_1, \ldots, a_n)$ for some L-structure A and some $a_1, \ldots, a_n \in A$.

Logic topology

Definition

Let T be a theory in L and let $\Gamma = \Gamma(x_1, \ldots, x_n)$ be a partial n-type in L.

- Γ *is consistent with* T if $T \cup \Gamma$ has a model.
- The set of all n-types that contain T is denoted by $S_n(T)$. These are exactly the n-types in L that are consistent with T.

The set $S_n(T)$ can be given the structure of a topological space, where the basic open sets are given by

$$[\varphi(x_1, \ldots, x_n)] = \{\Gamma(x_1, \ldots, x_n) \in S_n(T) : \varphi \in \Gamma\}.$$

This is called the *logic topology*.

Type spaces

Theorem

The space $S_n(T)$ with the logic topology is a totally disconnected, compact Hausdorff space. Its closed sets are the sets of the form

$$\{\Gamma \in S_n(T) : \Gamma' \subseteq \Gamma\}$$

where Γ' is a partial n-type. In fact, two partial n-types are equivalent over T iff they determine the same closed set. Furthermore, the clopen sets in the type space are precisely the ones of the form $[\varphi(x_1, \ldots, x_n)]$.