Diagrams

Definition

A *literal* is an atomic sentence or the negation of an atomic sentence. If M is a model in a language L, then the collection of L_M-literals true in M is called the *diagram* of M and written $\mathrm{Diag}(M)$. The collection of all L_M-sentences true in M is called the *elementary diagram* of M and written $\mathrm{ElDiag}(M)$.

Lemma

The following amount to the same thing:

- A model N of $\mathrm{Diag}(M)$.
- An embedding $h : M \to N$.

As do the following:

- A model N of $\mathrm{ElDiag}(M)$.
- An elementary embedding $h : M \to N$.

Upward Löwenheim-Skolem

Upward Löwenheim-Skolem

Suppose M is an infinite L-structure and κ is a cardinal number with $\kappa \geq |M|, |L|$. Then there is an elementary embedding $i : M \to N$ with $|N| = \kappa$.

Proof.

Let Γ be the elementary diagram of M and Δ be the set of sentences $\{c_i \neq c_j : i \neq j \in \kappa\}$ where the c_i are κ-many fresh constants. By the Compactness Theorem, the theory $\Gamma \cup \Delta$ has a model A; we have $|A| \geq \kappa$. By the downwards version A has an elementary substructure N of cardinality κ. So, since N is a model of Γ, there is an elementary embedding $i : M \to N$. $\square$

Characterisation universal theories

Theorem

T has a universal axiomatisation iff models of T are closed under substructures.

Proof.

Suppose T is a theory such that its models are closed under substructures. Let $T' = \{\varphi : T \models \varphi$ and φ is universal $\}$. Clearly, $T \models T'$. We need to prove the converse.

So suppose M is a model of T'. It sufffices to show that $T \cup \mathrm{Diag}(M)$ is consistent. Because once we do that, it will have a model N. But since N is a model of $\mathrm{Diag}(M)$, it will be an extension of M; and because N is a model of T and models of T are closed under substructures, M will be a model of T. □

Proof of claim

Claim

If $M \models T'$ where $T' = \{\varphi : T \models \varphi$ and φ is universal $\}$, then $T \cup \mathrm{Diag}(M)$ is consistent.

Proof.

Suppose not. Then, by the compactness theorem, there would be a finite set of literals $\psi_1, \ldots, \psi_n \in \mathrm{Diag}(M)$ which are inconsistent with T. Replace the constants from M in $\psi_1, \ldots, \psi_n$ by variables $x_1, \ldots, x_n$ and we obtain $\psi'_1, \ldots, \psi'_n$; because the constants from M do not appear in T, the theory T is already inconsistent with $\exists x_1, \ldots, x_n\, (\, \psi'_1 \wedge \ldots, \wedge \psi'_n\,)$. But then it follows that $T \models \neg \exists_1, \ldots, x_n\, (\, \psi'_1 \wedge \ldots \psi'_n\,)$ and $T \models \forall x_1, \ldots, x_n\, (\, \neg(\psi'_1 \wedge \ldots \psi'_n)\,)$, and hence $\forall x_1, \ldots, x_n\, (\, \neg(\psi'_1 \wedge \ldots \psi'_n)\,) \in T'$. But this contradicts the fact that M is a model of T'. □

Two exercises

Exercise

Prove: a theory has an existential axiomatisation iff its models are closed under extensions.

Exercise

For two L-structures A and B, we have $A \equiv B$ iff A and B have a common elementary extension.

Directed systems

See Chapters IV-VI in the lecture notes by Jaap van Oosten.

Definition

A partially ordered set $(K, \leq)$ is called *directed*, if K is non-empty and for any two elements $x, y \in K$ there is an element $z \in K$ such that $x \leq z$ and $y \leq z$.

Definition

A *directed system* of L-structures consists of a family $(M_k)_{k \in K}$ of L-structures indexed by K, together with homomorphisms $f_{kl} : M_k \to M_l$ for $k \leq l$. These homomorphisms should satisfy:

- f_{kk} is the identity homomorphism on M_k,
- if $k \leq l \leq m$, then $f_{km} = f_{lm} f_{kl}$.

If we have a directed system, then we can construct its *colimit*.

The colimit

First, we take the disjoint union of all the universes:

$$\sum_{k \in K} M_k = \{(k, a) : k \in K, a \in M_k\},$$

and then we define an equivalence relation on it:

$$(k, a) \sim (l, b) :\Leftrightarrow (\exists m \geq k, l)\, f_{km}(a) = f_{lm}(b).$$

Let M be the set of equivalence classes and denote the equivalence class of (k, a) by $[k, a]$.

The colimit, continued

M has an L-structure: we put

$$f^M([k_1, a_1], \ldots, [k_n, a_n]) \quad = \quad [k, f^{M_k}(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n)],$$

where k is an element $\geq k_1, \ldots, k_n$. (Check that this makes sense!)

And we put

$$R^M([k_1, a_1], \ldots, [k_n, a_n])$$

iff there is a $k \geq k_1, \ldots, k_n$ such that

$$(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n)) \in R^{M_k}.$$

In addition, we have maps $f_k : M_k \to M$ sending a to $[k, a]$.

Omnibus theorem

The following theorem collects the most important facts about colimits of filtered systems. Especially useful is part 5.

Theorem

1. All f_k are homomorphisms.
2. If $k \leq l$, then $f_l f_{kl} = f_k$.
3. If N is another L-structure for which there are homomorphisms $g_k : M_k \to N$ such that $g_l f_{kl} = g_k$ whenever $k \leq l$, then there is a unique homomorphisms $g : M \to N$ such that $g f_k = g_k$ for all $k \in K$ ("universal property").
4. If all maps f_{kl} are embeddings, then so are all f_k.
5. If all maps f_{kl} are elementary embeddings, then so are all f_k ("elementary system lemma").

Proof.

Exercise! □

Next goal

Our next big goal will be to prove:

Robinson's Consistency Theorem

Let L_1 and L_2 be two languages and $L = L_1 \cap L_2$. Suppose T_1 is an L_1-theory, T_2 an L_2-theory and both extend a complete L-theory T. If both T_1 and T_2 are consistent, then so is $T_1 \cup T_2$.

We first treat the special case where $L_1 \subseteq L_2$.

First lemma

Lemma

Let $L \subseteq L'$, A an L-structure and B an L'-structure. Suppose moreover $A \equiv B \restriction L$. Then there is an L'-structure C and a diagram of elementary embeddings (f in L and f' in L')

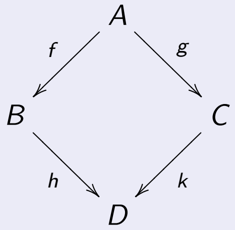
$$A \xrightarrow{f} C \xleftarrow{f'} B.$$

Proof. Consider $T = \mathrm{ElDiag}(A) \cup \mathrm{ElDiag}(B)$ (making sure we use different constants for the elements from A and B!). We need to show T has a model; so suppose T is inconsistent. Then, by Compactness, a finite subset of T has no model; taking conjunctions, we have sentences $\varphi(a_1, \ldots, a_n) \in \mathrm{ElDiag}(A)$ and $\psi(b_1, \ldots, b_m) \in \mathrm{ElDiag}(B)$ that are contradictory. But as the a_j do not occur in L_B, we must have that $B \models \neg\exists x_1, \ldots, x_n\, \varphi(x_1, \ldots, x_n)$. This contradicts $A \equiv B \restriction L$. $\square$

Second lemma

Lemma

Let $L \subseteq L'$ be languages, suppose A and B are L-structures and C is an L'-structure. Any pair of L-elementary embeddings $f : A \to B$ and $g : A \to C$ fit into a commuting square



where D is an L'-structure, h is an L-elementary embedding and k is an L'-elementary embedding.

Proof.

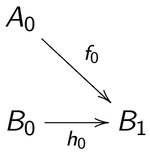
Without loss of generality we may assume that L contains constants for all elements of A. Then simply apply the first lemma. □

Robinson's consistency theorem

Theorem

Let L_1 and L_2 be two languages and $L = L_1 \cap L_2$. Suppose T_1 is an L_1-theory, T_2 an L_2-theory and both extend a complete L-theory T. If both T_1 and T_2 are consistent, then so is $T_1 \cup T_2$.

Proof. Let A_0 be a model of T_1 and B_0 be a model of T_2. Since T is complete, their reducts to L are elementary equivalent, so, by the first lemma, there is a diagram



with h_0 an L_2-elementary embedding and f_0 an L-elementary embedding. Now by applying the second lemma to f_0 and the identity on A, we obtain

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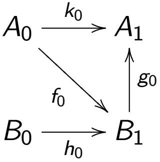
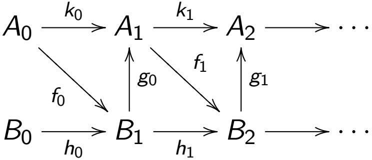
Robinson's consistency theorem, proof finished



where g_0 is L-elementary and k_0 is L_1-elementary. Continuing in this way we obtain a diagram



where the k_i are L_1-elementary, the f_i and g_i are L-elementary and the h_i are L_2-elementary. The colimit C of this directed system is both the colimit of the A_i and of the B_i. So A_0 and B_0 embed elementarily into C by the elementary systems lemma; hence C is a model of both T_1 and T_2, as desired. $\square$