Slides for a course on model theory

Benno van den Berg

Utrecht University, Fall 2012

Skolem theories

Definition

An L-theory T is a *Skolem theory* or *has built-in Skolem functions* if for every formula $\varphi(x_1, \ldots, x_n, y)$ there is a function symbol f such that

$$T \models \forall x_1, \ldots, x_n \left(\exists y \, \varphi(x_1, \ldots, x_n, y) \to \varphi(x_1, \ldots, x_n, f(x_1, \ldots, x_n) \right).$$

It is sufficient to require this for quantifier-free φ. (Exercise!)

Theorem

For every theory T in a language L there is a Skolem theory $T' \supseteq T$ in a language $L' \supseteq L$ with $|L'| \leq |L| + \aleph_0$ such that every model of T has an expansion to a model of T'.

Proof.

Write $L_0 = L$. Then let L_{n+1} be the language of Sk_{L_n} and put $L' = \bigcup L_n$ and $T' = T \cup \bigcup \mathrm{Sk}_{L_n}$. □

A theory T' as in the theorem is called a *skolemisation* of T.

Skolem hulls

Let M be a model of a Skolem theory T. Then for every subset $X \subseteq M$ the smallest subset of M containing X and closed under all the interpretations of the function symbols can be given the structure of a submodel of M. This is called the *Skolem hull* generated by X and denoted by $\langle X \rangle$.

Proposition

$\langle X \rangle$ is an elementary substructure of M.

Proof.

Exercise! (Hint: use Tarski-Vaught.) □

Downward Löwenheim-Skolem

Downward Löwenheim-Skolem

Suppose M is an L-structure and $X \subseteq M$. Then there is an elementary substructure N of M with $X \subseteq N$ and $|N| \leq |X| + |L| + \aleph_0$.

Proof.

Let T be the skolemisation of the empty theory in the language L and M' the expansion of M to a model of T. Then let N' be the Skolem hull generated by X. Then N' is an elementary substructure of M', and the reduct N of N' to the language L is an elementary substructure of M. □

Exercises

Proposition

A Skolem theory has a universal axiomatisation.

Proof.

Exercise! □

Proposition

A Skolem theory has quantifier-elimination.

Proof.

Exercise! □

Compactness Theorem

Definition

A theory T is *consistent* if every finite subset of T has a model.

Compactness Theorem

If a theory in a language L is consistent, then it has a model of cardinality $\leq |L| + \aleph_0$.

We will first prove this for universal theories.

Compactness theorem for universal theories

Compactness theorem for universal theories

If a universal theory in a language L is consistent, then it has a model of cardinality $\leq |L| + \aleph_0$.

Proof. Let T be a universal theory in a language L which is consistent. Without loss of generality, we may assume that L contains at least one constant: otherwise, simply add one to the language.

Let Δ the set of literals in the language L (a *literal* is an atomic sentence or its negation). Then the set

$$\{\Gamma \subseteq \Delta \ : \ T \cup \Gamma \text{ is consistent } \}$$

is partially ordered by inclusion. Moreover, every chain has an upper bound, so it contains a maximal element Γ_0 by Zorn's Lemma. For every atomic sentence we have either $\varphi \in \Gamma_0$ or $\neg\varphi \in \Gamma_0$.

Proof continued

We are now going to create a model M on the basis of the set Γ_0. Let $\mathcal{T}$ be the collection of terms in the language L. On $\mathcal{T}$ we can define a relation by:

$$s \sim t \Leftrightarrow s = t \in \Gamma_0.$$

This is an equivalence relation.

We can now define the interpretation of constants, function and relation symbols, as follows:

$$\begin{array}{rcl} c^M & = & [c], \\ f^M([t_1], \ldots, [t_n]) & = & [f(t_1, \ldots, t_n)], \\ R^M([t_1], \ldots, [t_n]) & \Leftrightarrow & R(t_1, \ldots, t_n) \in \Gamma_0. \end{array}$$

Check that this is well-defined! We have for every term t that $t^M = [t]$. Moreover, the set of literals true in M coincides precisely with Γ_0.

Proof finished

In order to finish the proof we need to show that M is a model of T. So consider a universal sentence $\forall x_1 \dots \forall x_n \, \psi(x_1, \dots, x_n)$ (ψ quantifier-free) that belongs to T. To show that it is valid in M we need to prove that for all terms $t_1, \dots, t_n$ we have

$$M \models \psi([t_1], \dots, [t_n]), \text{ or } M \models \psi(t_1, \dots, t_n).$$

Let S be the collection of all sentences all whose terms and relation symbols also occur in $\psi(t_1, \dots, t_n)$ and put $\Gamma_1 = \Gamma_0 \cap S$. Since there occur only finitely many terms and relation symbols in $\psi(t_1, \dots, t_n)$, the set Γ_1 is finite.

Because the set $T \cup \Gamma_0$ is consistent, there is a model N of $\{\forall x_1 \dots \forall x_n \, \psi(x_1, \dots, x_n)\} \cup \Gamma_1$. We have $N \models \varphi$ iff $\varphi \in \Gamma_1$ for all literals φ in S and hence $N \models \varphi$ iff $M \models \varphi$ for all quantifier-free sentences φ in S. So since we have $N \models \psi(t_1, \dots, t_n)$, we have $M \models \psi(t_1, \dots, t_n)$ as well. $\square$

Reduction

Lemma

Let T be a consistent theory in a language L. Then there is a language $L' \supseteq L$ with $|L'| \leq |L| + \aleph_0$ and a consistent universal theory T' in the language L' such that

1. every L-structures modelling T has an expansion to an L'-structure modelling T', and
2. every L-reduct of a model of T' is a model of T.

Proof.

Let L' be the language of Sk_L. By Skolem's theorem every sentence $\varphi \in T$ is equivalent modulo Sk_L to a quantifier-free sentence φ' in the language L'. Then let $T' = Sk_L \cup \{\varphi' : \varphi \in T\}$. □

General case

Compactness Theorem

If a theory in a language L is consistent, then it has a model of cardinality $\leq |L| + \aleph_0$.

Proof.

If T is a theory in language L which is consistent, then there is a universal theory T' in a richer language L' which is also consistent and is such that every L-reduct of a model of T' is a model of T. By the compactness theorem for universal theories, T' has a model M'. So the reduct of M' to L is a model of T. □

Diagrams

Definition

A *literal* is an atomic sentence or the negation of an atomic sentence. If M is a model in a language L, then the collection of L_M-literals true in M is called the *diagram* of M and written $\mathrm{Diag}(M)$. The collection of all L_M-sentences true in M is called the *elementary diagram* of M and written $\mathrm{ElDiag}(M)$.

Lemma

The following amount to the same thing:

- A model N of $\mathrm{Diag}(M)$.
- An embedding $h : M \to N$.

As do the following:

- A model N of $\mathrm{ElDiag}(M)$.
- An elementary embedding $h : M \to N$.