Second theorem

Today all theories are assumed to be *nice*.

Notation

Let A be an L-structure. If b is a tuple in A and B is any subset of A, we will write $\mathrm{tp}_A(b/B)$ for the type in L_B realized by b.

Theorem

Assume T is an ω-stable theory, and suppose $A \models T$ and $C \subseteq A$. If A is uncountable and $|C| < |A|$, then there is a nonconstant sequence of indiscernibles in $(A, a)_{a \in C}$.

Proof.

We may assume C is infinite. Write $\lambda = |C|$. The formula $x = x$ is satisfied by $> \lambda$ many elements, so choose an L_A-formula $\varphi(x)$ that is satisfied by $> \lambda$ many elements and has minimum possible Morley rank and degree; say these are (α, d). Note that $\alpha > 0$ since $\varphi(x)$ is satisfied by infinitely many elements. By adding finitely many elements to C we may assume that $\varphi(x)$ is an L_C-formula. □

Second theorem, proof continued

Proof.

We will construct a sequence $(a_k : k \in \mathbb{N})$ of elements of A that satisfy $\varphi(x)$ and such that Morley rank and degree of $\mathrm{tp}_A(a_k/C \cup \{a_0, \ldots, a_{k-1}\})$ is exactly (α, d).

First we claim that there is an a_0 with this property. For if no such element would exist, we would have that Morley rank and degree of $\mathrm{tp}_A(a/C)$ is $< (\alpha, d)$ for all $a \in A$ satisfying $\varphi(x)$. So each $a \in A$ which satisfies $\varphi(x)$ also satisfies an L_C-formula $\psi_a(x)$ with Morley degree and rank $< (\alpha, d)$. But since there are at most λ many L_C-formulas and more than λ many a satisfying $\varphi(x)$, there must be a formula with Morley rank and degree $< (\alpha, d)$ satisfied by $> \lambda$ many a. Contradiction! The construction of a_k given $a_0, \ldots, a_{k-1}$ is similar. So the result follows from the following technical lemma. □

Technical lemma

Lemma

Assume T is ω-stable and suppose $A \models T$ and $C \subseteq A$. Let $\varphi(x)$ be a ranked L_C-formula, and set $(\alpha, d) = (\mathrm{RM}(\varphi(x)), dM(\varphi(x)))$. Suppose $(a_k : k \in \mathbb{N})$ is a sequence of tuples and write $p_k(x) = \mathrm{tp}_A(a_k / C \cup \{a_0, \ldots, a_{k-1}\})$. If $A \models \varphi(a_k)$ and $(\mathrm{RM}(p_k(x)), dM(p_k(x))) = (\alpha, d)$, then $(a_k : k \in \mathbb{N})$ is an indiscernible sequence in $(A, a)_{a \in C}$.

Proof.

Exercise! Hint: Prove by induction on n that whenever $i_0 < \ldots < i_n$, then $\mathrm{tp}(a_{i_0}, \ldots, a_{i_n}/C) = \mathrm{tp}(a_0, \ldots, a_n/C)$ and use the lemma on types and Morley rank and degree. □

Third goal

Recall that the third goal was:

Theorem

Assume T is ω-stable. If $A \models T$ and $C \subseteq A$, then there exists $B \preceq A$ such that $C \subseteq B$ and B is atomic over C.

We do this in two steps: first we show that we can find such a B where B is *constructible* over C; and then we show that constructible extensions have to be atomic.

Definition

Let A be an L-structure and $C \subseteq A$. We say that A *is constructible over* C if there is an ordinal γ and an enumeration $A = (a_\alpha : \alpha < \gamma)$ such that each a_α is atomic over $C \cup A_\alpha$, where $A_\alpha = \{a_\mu : \mu < \alpha\}$.

Existence constructible extensions

Theorem

Assume T is ω-stable. If $A \models T$ and $C \subseteq A$, then there exists $B \preceq A$ such that $C \subseteq B$ and B is constructible over C.

Proof.

T is totally transcendental, so if B is a subset of a model A of T, then $\mathrm{Th}(A_B)$ has no binary tree of consistent formulas. So isolated types in $\mathrm{Th}(A_B)$ are dense.

Now use Zorn's Lemma to find a maximal construction $(a_\alpha)_{a<\lambda}$ which cannot be prolonged by an element $a_\lambda \in M$. Clearly C is contained in A_λ. We show that A_λ is the universe of an elementary substructure by using the Tarski-Vaught Test. So assume $\varphi(x)$ is an L_{A_λ}-formula and $A \models \exists x\, \varphi(x)$. Since isolated types over A_λ are dense, there is an isolated $p(x) \in S(A_\lambda)$ with $\varphi(x) \in p(x)$. Let b be a realisation of $p(x)$ in A. If $b \notin A_\lambda$, then we could prolong our construction by $a_\lambda = b$; thus $b \in A_\lambda$ and $\varphi(x)$ is realised in A_λ. □

Useful lemma

Lemma

Let a and b be two finite tuples of elements of a structure M. Then $\mathrm{tp}(ab)$ is atomic if and only if $\mathrm{tp}(a/b)$ and $\mathrm{tp}(b)$ are atomic.

Proof.

First assume that $\varphi(x, y)$ isolates $\mathrm{tp}(a, b)$. Then $\varphi(x, b)$ isolates $\mathrm{tp}(a/b)$ and we claim $\exists x\, \varphi(x, y)$ isolates $p(y) = \mathrm{tp}(b)$: we have $\exists x\, \varphi(x, y) \in p(y)$ and if $\sigma(y) \in p(y)$, then $M \models \forall x, y\, (\, \varphi(x, y) \to \sigma(y)\,)$ and hence $M \models \forall y\, (\, \exists x\, \varphi(x, y) \to \sigma(y)\,)$.

Conversely, suppose $\rho(x, b)$ isolates $\mathrm{tp}(a/b)$ and $\sigma(y)$ isolates $p(y) = \mathrm{tp}(b)$. Then $\rho(x, y) \wedge \sigma(y)$ isolates $\mathrm{tp}(a, b)$. For if $\varphi(x, y) \in \mathrm{tp}(a, b)$, then $\varphi(x, b)$ belongs to $\mathrm{tp}(a/b)$ and $M \models \forall x\, (\, \rho(x, b) \to \varphi(x, b)\,)$. Hence $\forall x\, (\, \rho(x, y) \to \varphi(x, y)) \in p(y)$ and so it follows that $M \models \forall y\, (\, \sigma(y) \to \forall x\, (\, \rho(x, y) \to \varphi(x, y)\,)$. Thus $M \models \forall x, y\, (\, \rho(x, y) \wedge \sigma(y) \to \varphi(x, y)\,)$. □

Constructible extensions are atomic

Lemma

Constructible extensions are atomic.

Proof.

Let M_0 be a constructible extension of A and let $\overline{a}$ be a tuple from M_0. We have to show that $\overline{a}$ is atomic over A. We can clearly assume that the elements of $\overline{a}$ are pairwise distinct and do not belong to A. We can permute the elements of $\overline{a}$ so that

$$\overline{a} = a_\alpha \overline{b}$$

for some tuple $\overline{b} \in A_\alpha$. Let $\varphi(x, \overline{c})$ be an $L(A_\alpha)$-formula which is complete over A_α and satisfied by a_α. The a_α is also atomic over $A \cup \{\overline{b}\overline{c}\}$. Using induction, we know that $\overline{b}\overline{c}$ is atomic over A. So by the previous lemma $a_\alpha \overline{b}\overline{c}$ and $\overline{a} = a_\alpha \overline{b}$ are atomic over A. □

κ-categoricity and saturation

Theorem

A theory T is κ-categorical if and only if all models of cardinality κ are κ-saturated.

For the proof we need a lemma:

Lemma

If T is κ-stable, then for all regular $\lambda \leq \kappa$ there is a model of cardinality κ which is λ-saturated.

Proof.

We constuct a sequence $(M_\alpha : \alpha \in \lambda)$ of models of T of cardinality κ: we start with any model M_0 of cardinality κ of T; at limit stages we take the colimit and at successor stages we take a model $M_{\alpha+1}$ which realises all types in $S(M_\alpha)$. This we can do with a model of cardinality κ since $|S(M_\alpha)| \leq \kappa$. The colimit of the entire chain will be λ-saturated. $\square$

κ-categoricity and saturation: proof

Theorem

A theory T is κ-categorical if and only if all models of cardinality κ are κ-saturated.

Proof.

Note that we already proved this result for $\kappa = \omega$ and that we also know that any two κ-saturated models of cardinality κ are isomorphic. So we only need to show that if T is κ-categorical for some uncountable cardinal κ, then all models of cardinality κ are κ-saturated.

But then T is ω-stable, hence totally transcendental, hence κ-stable. So by the lemma the unique model of T of cardinality κ is μ^+-saturated for all $\mu < \kappa$. So this model is κ-saturated. □

A theorem implying Morley's theorem

So Morley's Theorem will follow from:

Theorem

Suppose T is ω-stable and assume κ is an uncountable cardinal and that every model of T of cardinality κ is κ-saturated. Then every uncountable model of T is saturated.

Proof.

Suppose T is ω-stable and T has a model of cardinality λ that is not λ-saturated. (Goal is to construct a model of cardinality κ that is not κ-saturated.) So there is a subset C of A of cardinality $< \lambda$ and a type $p(x)$ over C such that $p(x)$ is consistent with $\mathrm{Th}((A, a)_{a \in C})$ but not realized in $(A, a)_{a \in C}$. We know that there is a nonconstant sequence $(a_k : k \in \mathbb{N})$ of indiscernibles in $(A, a)_{a \in C}$ (second goal). Write $I = \{a_k : k \in \mathbb{N}\}$ and note that (*): *for each $L(C \cup I)$-formula $\varphi(x)$ that is satisfiable in $(A, a)_{a \in C \cup I}$ there exists $\psi(x) \in p(x)$ such that $\varphi(x) \wedge \neg\psi(x)$ is satisfiable in $(A, a)_{a \in C \cup I}$.* (For otherwise $p(x)$ would be realized in $(A, a)_{a \in C}$.) □

A theorem implying Morley's theorem, proof continued

Proof.

We have (*): *for each* $L(C \cup I)$*-formula* $\varphi(x)$ *that is satisfiable in* $(A, a)_{a \in C \cup I}$ *there exists* $\psi(x) \in p(x)$ *such that* $\varphi(x) \wedge \neg\psi(x)$ *is satisfiable in* $(A, a)_{a \in C \cup I}$.

Let C_0 be any countable subset of C. For each $L(C_0 \cup I)$ formula $\varphi(x)$ that is satisfiable in $(A, a)_{a \in C_0 \cup I}$ let ψ_φ be one of the formulas satisfying (*) for φ. Since $C_0 \cup I$ is countable, there is a countable set C_1 such that $C_0 \subseteq C_1 \subseteq C$ and such that the parameters of ψ_φ are in C_1. Continuing in this way to create sets C_k, let $C' = \bigcup\{C_k : k \in \mathbb{N}\}$. Let $p'(x)$ be restriction of $p(x)$ to C'. We have (**): *for each* $L(C' \cup I)$*-formula* $\varphi(x)$ *that is satisfiable in* $(A, a)_{a \in C' \cup I}$ *there exists* $\psi(x) \in p'(x)$ *such that* $\varphi(x) \wedge \neg\psi(x)$ *is satisfiable in* $(A, a)_{a \in C' \cup I}$. Note also that $(a_k : k \in \mathbb{N})$ is a sequence of indiscernibles in $(A, a)_{a \in C'}$. □

A theorem implying Morley's theorem, proof continued

Proof.

By the Standard Lemma there is a model B of $\mathrm{Th}((A, a)_{a \in C'})$ that contains a family $(b_\alpha : \alpha < \kappa)$ realising the Ehrenfeucht-Mostowski type of $(a_k : k \in \mathbb{N})$. We may assume this model is of the form $(B, a)_{a \in C'}$. Using the Third Goal we know that there is an elementary substructure B' of B which is atomic over $C' \cup \{b_\alpha : \alpha < \kappa\}$.

The proof will be finished once we show that $p'(x)$ is not realised in $(B', a)_{a \in C'}$. For then the downward Löwenheim-Skolem Theorem implies that B' has an elementary substructure B'' of cardinality κ which contains C'. Then B'' is a model of cardinality κ which is not κ-saturated. (In fact, it is not even ω_1-saturated.) □

A theorem implying Morley's theorem, proof finished

Claim

The type $p'(x)$ is not realised in $(B', a)_{a \in C'}$.

Proof.

Recall that we have (**): *for each $L(C' \cup I)$-formula $\varphi(x)$ that is satisfiable in $(A, a)_{a \in C' \cup I}$ there exists $\psi(x) \in p'(x)$ such that $\varphi(x) \wedge \neg\psi(x)$ is satisfiable in $(A, a)_{a \in C' \cup I}$.*

So suppose $p'(x)$ is realised in $(B', a)_{a \in C'}$ by some tuple b. We have that $\mathrm{tp}_{B'}(b/C' \cup \{b_\alpha : \alpha < \kappa\})$ is isolated so it contains a complete formula $\varphi(x, b_{\alpha_0}, \ldots, b_{\alpha_n})$. So we have that $\varphi(x, b_{\alpha_0}, \ldots, b_{\alpha_n}) \rightarrow \psi(x)$ holds in B' for every $\psi(x) \in p'(x)$. But since $b_{\alpha_0}, \ldots, b_{\alpha_n}$ and $a_0, \ldots, a_n$ realize the same Ehrenfeucht-Mostowski type over C', we have that $\varphi(x, a_0, \ldots, a_n) \rightarrow \psi(x)$ is valid in A for each formula $\psi(x) \in p'(x)$. But that contradicts (**). □

Morley's Theorem

Morley's Theorem

If a countable theory T is λ-categorical for an uncountable cardinal λ, then it is λ-categorical for all uncountable cardinal λ.

End of the course. And Merry Christmas and Happy New Year!