Existence countable saturated models

Convention

Let us say a theory is *nice* if it

- is complete,
- and formulated in a countable language,
- and has infinite models.

Definition

A theory T is *small* if all $S_n(T)$ are at most countable.

Theorem

A nice theory is small iff it has a countable ω-saturated model.

Proof.

$\Leftarrow$: If T is complete and has a countable ω-saturated model, then every type consistent with T is realized in that model. So there are at most countable many n-types for any n. (For $\Rightarrow$ see next page.) □

Proof finished

Theorem

A nice theory is small iff it has a countable ω-saturated model.

Proof.

$\Rightarrow$: We know that a model A can be elementarily embedded in a model B which realizes all types with parameters from A that are finitely satisfied in A. From the proof of that result we see that if A is a countable and there are at most countably many n-types with a finite set of parameters from A, then all of these types can be realized in a *countable* elementary extension B. Building an ω-chain by repeatedly applying this result and then taking the colimit, we see that A can be embedded in a countable ω-saturated elementary extension. So if A is a countable model of T, we obtain the desired result. □

Omitting types theorem

Definition

Let T be an L-theory and $p(x)$ be a partial type. Then $p(x)$ is *isolated in* T if there is a formula $\varphi(x)$ such that $\exists x\, \varphi(x)$ is consistent with T and

$$T \models \varphi(x) \to \sigma(x)$$

for all $\sigma(x) \in p(x)$.

Omitting types theorem

Let T be a consistent theory in a countable language. If a partial type $p(x)$ is not isolated in T, then there is a countable model of T which omits $p(x)$.

Lemma

Lemma

Suppose T is a consistent theory in a language L and C is a set of constants in L. If for any formula $\psi(x)$ in the language L there is a constant $c \in C$ such that

$$T \models \exists x\, \psi(x) \rightarrow \psi(c),$$

then T has a model whose universe consists entirely of interpretations of elements of C.

Proof.

Extend T to a maximally consistent theory using the Lemma on page 4 of the slides for week 2 and then apply the Lemma on page 3 of the slides for week 2. $\square$

Omitting types theorem, proof

Omitting types theorem

Let T be a consistent theory in a countable language. If a partial type $p(x)$ is not isolated in T, then there is a countable model of T which omits $p(x)$.

Proof.

Let $C = \{c_i\,;\, i \in \mathbb{N}\}$ be a countable collection of fresh constants and L_C be the language L extending with these constants. Let $\{\psi_i(x) : i \in \mathbb{N}\}$ be an enumeration of the formulas with one free variable in the language L_C. We will now inductively create a sequence of sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$. The idea is to apply to previous lemma to $T \cup \{\varphi_0, \varphi_1, \ldots\}$.

If $n = 2i$, we take a fresh constant $c \in C$ (one that does not occur in φ_m with $m < n$) and put

$$\varphi_n = \exists x \psi_i(x) \to \psi(c).$$

This makes sure we can create a model from the constants in C. □

Omitting types theorem, proof finished

Proof.

If $n = 2i + 1$ we make sure that c_i omits $p(x)$, as follows. Consider $\delta = \bigwedge_{m<n} \varphi_m$. δ is really of the form $\delta(c_i, \overline{c})$ where $\overline{c}$ is a sequence of constants not containing c_i. Since $p(x)$ is not isolated, there must be a formula $\sigma(x) \in p(x)$ such that $T \not\models \exists \overline{y}\, \delta(x, \overline{y}) \to \sigma(x)$; in other words, such that $T \cup \{\exists \overline{y}\, \delta(x, y)\} \cup \{\neg \sigma(x)\}$ is consistent. Put $\varphi_{2n} = \neg \sigma(c_i)$.

The proof is now finished by showing by induction that each $T \cup \{\varphi_0, \ldots, \varphi_n\}$ is consistent and then applying the previous lemma. □

Remark

Note that for any theory T we have:

Proposition

The following are equivalent: (1) all n-types are isolated; (2) every $S_n(T)$ is finite; (3) for every n there are only finite many formulas $\varphi(x_1, \ldots, x_n)$ up to equivalence relative to T.

Proof.

(1) $\Leftrightarrow$ (2) holds because $S_n(T)$ is a compact Hausdorff space.
(2) $\Rightarrow$ (3): If there are only finitely many types, then each of these isolated, so there are formulas $\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$ "isolating" all these types with $T \models \bigvee_i \psi_i$. But then every formula $\varphi(x_1, \ldots, x_n)$ is equivalent to the disjunction of the ψ_i of which it is a consequence.
(3) $\Rightarrow$ (2): If every formula $\varphi(x_1, \ldots, x_n)$ is equivalent modulo T to one of $\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)$, then every n-type is completely determined by saying which ψ_i it does and which it does not contain. □

Ryll-Nardzewski Theorem

Theorem (Ryll-Nardzewski)

For a nice theory T the following are equivalent:

1. T is ω-categorical;
2. all n-types are isolated;
3. all models of T are ω-saturated;
4. all countable models of T are ω-saturated.

Proof.

(1) $\Rightarrow$ (2): If T contains a non-isolated type then there is a model where it is realized and a model where it is not realized (by the Omitting Types Theorem). (2) $\Rightarrow$ (3): If all $n+1$-types are isolated, then every 1-type with n parameters from a model is isolated, hence generated by a single formula. So if such a type is finitely satisfiable in a model, that formula can be satisfied there and then the entire type is realised. (3) $\Rightarrow$ (4) is obvious. (4) $\Rightarrow$ (1): Because elementarily equivalent κ-saturated models of cardinality κ are always isomorphic. □

Vaught's Theorem

Corollary

If A is a model and $a_1, \ldots, a_n$ are elements from A, then $\mathrm{Th}(A)$ is ω-categorical iff $\mathrm{Th}(A, a_1, \ldots, a_n)$ is ω-categorical.

Theorem (Vaught)

A nice theory cannot have exactly two countable models (up to isomorphism).

Proof.

Let T be a nice theory. Without loss of generality we may assume that T is small (why?) and not ω-categorical. We will now show that T has at least three models. First of all, there is a countable ω-saturated model A. In addition, there is a non-isolated type p which is omitted in some model B. Of course, it is realized in A by some tuple $\overline{a}$. Since $\mathrm{Th}(A, \overline{a})$ is not ω-categorical, it has a model different from A. Since this model realizes p, it must be different from B as well. □

Prime and atomic models

Definition

Let T be a nice theory.

- A model M of T is called *prime* if it can be elementarily embedded into any model of T.
- A model M of T is called *atomic* if it only realises isolated types (or, put differently, omits all non-isolated types) in $S_n(T)$.

Theorem

A model of a nice theory T is prime iff it is countable and atomic.

Proof.

$\Rightarrow$: Because T is nice it has countable models and non-isolated types can be omitted. For $\Leftarrow$ see the next page. □

Proof continued

Theorem

A model of a nice theory T is prime iff it is countable and atomic.

Proof.

$\Leftarrow$: Let A be a countable and atomic model of a nice theory T and M be any other model of T. Let $\{a_1, a_2, \ldots\}$ be an enumeration of A; by induction on n we will construct an increasing sequence of elementary maps $f_n : \{a_1, \ldots, a_n\} \to M$. We start with $f_0 = \emptyset$, which is elementary as A and M are elementarily equivalent. (They are both models of a complete theory T.)

Suppose f_n has been constructed. The type of $a_1, \ldots, a_{n+1}$ in A is isolated, hence generated by a single formula $\varphi(x_1, \ldots, x_{n+1})$. In particular, $A \models \exists x_{n+1}\, \varphi(a_1, \ldots, a_n, x_{n+1})$, and since f_n is elementary, $M \models \exists x_{n+1}\, \varphi(f_n(a_1), \ldots, f_n(a_n), x_{n+1})$. So choose $m \in M$ such that $M \models \varphi(f_n(a_1), \ldots, f_n(a_n), m)$ and put $f(a_{n+1}) = m$. □

Existence prime models

Theorem

All prime models of a nice theory T are isomorphic. In addition, they are strongly ω-homogeneous.

Proof.

By the familiar back-and-forth techniques. (Exercise!) □

Theorem

A nice theory T has a prime model iff the isolated n-types are dense in $S_n(T)$ for all n.

Remark

Let us call a formula $\varphi(\overline{x})$ *complete* in T if it generates an isolated type in $S_n(T)$: that is, it is consistent and for any other formula $\psi(\overline{x})$ we have either $T \models \varphi(\overline{x}) \to \psi(\overline{x})$ or $T \models \varphi(\overline{x}) \to \neg\psi(\overline{x})$. Then n-types are dense iff every consistent formula $\varphi(\overline{x})$ follows from some complete formula.

Existence prime models, proof

Theorem

A nice theory T has a prime model iff the isolated n-types are dense in $S_n(T)$ for all n.

Proof.

$\Rightarrow$: Let A be a prime model of T. Because a consistent formula $\varphi(\bar{x})$ is realised in *all* models of T, it is realized in A as well, by $\bar{a}$ say. Since A is atomic, $\varphi(\bar{x})$ belongs to the isolated type $\mathrm{tp}_A(\bar{a})$.
$\Leftarrow$: Note that a structure A is atomic iff the sets

$$p_n(x_1, \ldots, x_n) = \{\, \neg\varphi(x_1, \ldots, x_n) \,:\, \varphi \text{ is complete} \,\}$$

are omitted in A. So it suffices to show that the p_n are not isolated (by the generalised omitting types theorem). But that holds iff for any consistent $\psi(\bar{x})$ there is a complete formula $\varphi(\bar{x})$ such that $T \not\models \psi(\bar{x}) \rightarrow \neg\varphi(\bar{x})$. As $\varphi(\bar{x})$ is complete, this is equivalent to $T \models \varphi(\bar{x}) \rightarrow \psi(x)$. So the Σ_n are not isolated iff isolated types are dense. □

Binary trees of formulas

Definition

Let $\{0,1\}^*$ be the set of finite sequences consisting of zeros and ones. A *binary tree* of formulas in variables $\overline{x} = x_1, \ldots, x_n$ (in T) is a collection $\{\varphi_s(\overline{x}) : s \in \{0,1\}^*\}$ such that

- $T \models \big(\varphi_{s0}(\overline{x}) \vee \varphi_{s1}(\overline{x})) \rightarrow \varphi_s(\overline{x})\big)$.
- $T \models \neg\big(\varphi_{s0}(\overline{x}) \wedge \varphi_{s1}(\overline{x})\big)$.

Theorem

The following are equivalent for a nice theory T:

(1) $|S_n(T)| < 2^\omega$.

(2) There is no binary tree of consistent formulas in $x_1, \ldots, x_n$.

(3) $|S_n(T)| \leq \omega$.

Clearly, if $\{\varphi_s(\overline{x}) : s \in \{0,1\}^*\}$ is a binary tree of consistent formulas, $\{\varphi_s : s \subseteq \alpha\}$ is consistent for every $\alpha : \mathbb{N} \rightarrow \{0,1\}$. This shows (1) $\Rightarrow$ (2). As (3) $\Rightarrow$ (1) is obvious, it remains to show (2) $\Rightarrow$ (3).

A lemma

Lemma

Let T be a nice theory. If $|S_n(T)| > \omega$, then there is a binary tree of consistent formulas in $x_1, \ldots, x_n$.

Proof.

Suppose $|S_n(T)| > \omega$. This implies, since the language of T is countable, that there is a formula $\varphi(\overline{x})$ such that $|[\varphi]| > \omega$. The lemma will now follow from the following *claim*: If $|[\varphi]| > \omega$, then there is a formula $\psi(\overline{x})$ such that $|[\varphi \wedge \psi]| > \omega$ and $|[\varphi \wedge \neg\psi]| > \omega$. Suppose not.
Then $p(\overline{x}) = \{\psi(\overline{x}) : |[\varphi \wedge \psi]| > \omega\}$ contains a formula $\psi(\overline{x})$ or its negation, but not both, and is closed under logical consequence: so it is a complete type. If $\psi \notin p$, then $|[\varphi \wedge \psi]| \leq \omega$. In addition, the language is countable, so

$$[\varphi] = \bigcup_{\psi \notin p} [\varphi \wedge \psi] \cup \{p\}$$

is a countable union of countable sets and hence countable, contradicting our choice of φ. □

Small theories have prime models

Corollary

If T is nice and $|S_n(T)| < 2^\omega$ for all n, then T is small.

Corollary

If T is nice and small, then isolated types are dense. So T has a prime model.

Proof.

If isolated types are not dense, then there is a consistent $\varphi(\bar{x})$ which is not a consequence of a complete formula. Call such a formula *perfect*. Since perfect formulas are not complete, they can be "decomposed" into two consistent formulas which are jointly inconsistent. These have to be perfect as well, leading to a binary tree of consistent formulas. □