Section 1

Basic definitions

Language

A *language* or *signature* consists of:

1. constants.
2. function symbols.
3. relation symbols.

Once and for all, we fix a countably infinite set of variables. The terms are the smallest set such that:

1. all constants are terms.
2. all variables are terms.
3. if $t_1, \ldots, t_n$ are terms and f is an n-ary function symbol, then also $f(t_1, \ldots, t_n)$ is a term.

Terms which do not contain any variables are called *closed*.

Formulas and sentences

The *atomic formulas* are:

1. $s = t$, where s and t are terms.
2. $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are terms and P is a predicate symbol.

The set of *formulas* is the smallest set which:

1. contains the atomic formulas.
2. is closed under the propositional connectives $\wedge, \vee, \rightarrow, \neg$.
3. contains $\exists x\, \varphi$ and $\forall x\, \varphi$, if φ is a formula.

A formula which does not contain any quantifiers is called *quantifier-free*. A *sentence* is a formula which does not contain any free variables. A set of sentences is called a *theory*.

Convention: If we write $\varphi(x_1, \ldots, x_n)$, this is supposed to mean: φ is a formula and its free variables are contained in $\{x_1, \ldots, x_n\}$.

Models

A *structure* or *model* M in a language L consists of:

1. a non-empty set M (the *domain* or the *universe*).
2. interpretations $c^M \in M$ of all the constants in L,
3. interpretations $f^M : M^n \to M$ of all function symbols in L,
4. interpretations $R^M \subseteq M^n$ of all relation symbols in L.

The interpretation can then be extended to all terms in the language:

$$f(t_1, \ldots, t_n)^M = f^M(t_1^M, \ldots, f_n^M).$$

If $A \subseteq M$, then we will write L_A for the language obtained by adding to L fresh constants $\{c_a : a \in A\}$. In this case M is also an L_A-structure with c_a to be interpreted as a. We will often just write a instead of c_a.

Tarski's truth definition

Validity or truth

If M is a model and φ is a sentence in the language L_M, then:

- $M \models s = t$ iff $s^M = t^M$;
- $M \models P(t_1, \ldots, t_n)$ iff $(t_1, \ldots, t_n) \in P^M$;
- $M \models \varphi \wedge \psi$ iff $M \models \varphi$ and $M \models \psi$;
- $M \models \varphi \vee \psi$ iff $M \models \varphi$ or $M \models \psi$;
- $M \models \varphi \rightarrow \psi$ iff $M \models \varphi$ implies $M \models \psi$;
- $M \models \neg\varphi$ iff not $M \models \varphi$;
- $M \models \exists x\, \varphi(x)$ iff there is an $m \in M$ such that $M \models \varphi(m)$;
- $M \models \forall x\, \varphi(x)$ iff for all $m \in M$ we have $M \models \varphi(m)$.

Semantic implication

Definition

If M is a model in a language L, then $\mathrm{Th}(M)$ is the collection L-sentences true in M. If N is another model in the language L, then we write $M \equiv N$ and call M and N *elementarily equivalent*, whenever $\mathrm{Th}(M) = \mathrm{Th}(N)$.

Definition

Let Γ and Δ be theories. If $M \models \varphi$ for all $\varphi \in \Gamma$, then M is called a *model* of Γ. We will write $\Gamma \models \Delta$ if every model of Γ is a model of Δ as well. We write $\Gamma \models \varphi$ for $\Gamma \models \{\varphi\}$, et cetera.

Expansions and reducts

If $L \subseteq L'$ and M is an L'-structure, then we can obtain an L-structure N by taking the universe of M and forgetting the interpretations of the symbols which do not occur in L. In that case, M is an *expansion* of N and N is the *L-reduct* of M.

Lemma

If $L \subseteq L'$ and M is an L'-structure and N is its L-reduct, then we have $N \models \varphi(m_1, \ldots, m_n)$ iff $M \models \varphi(m_1, \ldots, m_n)$ for all formulas $\varphi(x_1, \ldots, x_n)$ in the language L.

Homomorphisms

Let M and N be two L-structures. A *homomorphism* $h : M \to N$ is a function $h : M \to N$ such that:

1. $h(c^M) = c^N$ for all constants c in L;
2. $h(f^M(m_1, \ldots, m_n)) = f^N(h(m_1), \ldots, h(m_n))$ for all function symbols f in L and elements $m_1, \ldots, m_n \in M$;
3. $(m_1, \ldots, m_n) \in R^M$ implies $(h(m_1), \ldots, h(m_n)) \in R^N$.

A homomorphism which is bijective and whose inverse f^{-1} is also a homomorphism is called an *isomorphism*. If an isomorphism exists between structures M and N, then M and N are called *isomorphic*. An isomorphism from a structure to itself is called an *automorphism*.

Embeddings

A homomorphism $h : M \to N$ is an *embedding* if

1. h is injective;
2. $(h(m_1), \ldots, h(m_n)) \in R^N$ implies $(m_1, \ldots, m_n) \in R^M$.

Lemma

The following are equivalent for a homomorphism $h : M \to N$:

1. it is an embedding.
2. $M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_n))$ for all $m_1, \ldots, m_n \in M$ and atomic formulas $\varphi(x_1, \ldots, x_n)$.
3. $M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_n))$ for all $m_1, \ldots, m_n \in M$ and quantifier-free formulas $\varphi(x_1, \ldots, x_n)$.

If M and N are two models and the inclusion $M \subseteq N$ is an embedding, then M is a *substructure* of N and N is an *extension* of M.

Elementary embeddings

An embedding is called *elementary*, if

$$M \models \varphi(m_1, \ldots, m_n) \Leftrightarrow N \models \varphi(h(m_1), \ldots, h(m_n))$$

for all $m_1, \ldots, m_n \in M$ and *all* formulas $\varphi(x_1, \ldots, x_n)$.

Lemma

If h is an isomorphism, then h is an elementary embedding. If there is an elementary embedding $h : M \to N$, then $M \equiv N$.

Tarski-Vaught Test

An embedding $h : M \to N$ is elementary if and only if for any L_M-formula $\varphi(x)$: if $N \models \exists x\, \varphi(x)$, then there is an element $m \in M$ such that $N \models \varphi(h(m))$.

Cardinality of model and language

Definition

The *cardinality* of a model is the cardinality of its underlying domain. The cardinality of a language L is the sums of the cardinalities of its sets of constants, function symbols and relation symbols.

I will write $|X|$ for the cardinality of the set X, $|M|$ for the cardinality of the model M and $|L|$ for the cardinality of the language L.

Downward Löwenheim-Skolem

Downward Löwenheim-Skolem

Suppose M is an L-structure and $X \subseteq M$. Then there is an elementary substructure N of M with $X \subseteq N$ and $|N| \leq |X| + |L| + \aleph_0$.

Proof.

We construct N as $\bigcup_{i \in \mathbb{N}} N_i$ where the N_i are defined inductively as follows: $N_0 = X$, while

- if i is even, then N_{i+1} is obtained from N_i by adding the interpretations of the constants and closing under f^M for every function symbol f.
- if i is odd, we look at all L_{N_i}-sentences of the form $\exists x\, \varphi(x)$. If such a sentence is true in M, then we pick a witness $n \in M$ such that $M \models \varphi(n)$ and put it in N_{i+1}.

Then the first item guarantees that N is a substructure, while the second item ensures that it is an elementary substructure (using the Tarski-Vaught test). □

Section 2

New models from old

Directed systems

Definition

A partially ordered set $(K, \leq)$ is called *directed*, if K is non-empty and for any two elements $x, y \in K$ there is an element $z \in K$ such that $x \leq z$ and $y \leq z$. It is a *chain*, if K is non-empty and for any two elements $x, y \in K$ either $x \leq y$ or $y \leq x$.

Clearly, chains are directed.

Definition

A *directed system* of L-structures consists of a family $(M_k)_{k \in K}$ of L-structures indexed by K, together with homomorphisms $f_{kl} : M_k \to M_l$ for $k \leq l$. These homomorphisms should satisfy:

- f_{kk} is the identity homomorphism on M_k,
- if $k \leq l \leq m$, then $f_{km} = f_{lm} f_{kl}$.

If we have a directed system, then we can construct its *colimit*.

The colimit

First, we take the disjoint union of all the universes:

$$\sum_{k \in K} M_k = \{(k, a) \,:\, k \in K, a \in M_k\},$$

and then we define an equivalence relation on it:

$$(k, a) \sim (l, b) :\Leftrightarrow (\exists m \geq k, l)\, f_{km}(a) = f_{lm}(b).$$

Let M be the set of equivalence classes and denote the equivalence class of (k, a) by $[k, a]$.

The colimit, continued

M has an L-structure: we put

$$f^M([k_1, a_1], \ldots, [k_n, a_n]) \quad = \quad [k, f^{M_k}(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n)],$$

where k is an element $\geq k_1, \ldots, k_n$. (Check that this makes sense!)

And we put

$$R^M([k_1, a_1], \ldots, [k_n, a_n])$$

iff there is a $k \geq k_1, \ldots, k_n$ such that

$$(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n)) \in R^{M_k}.$$

In addition, we have maps $f_k : M_k \to M$ sending a to $[k, a]$.

Omnibus theorem

The following theorem collects the most important facts about colimits of directed systems. Especially useful is part 5.

Theorem

1. All f_k are homomorphisms.
2. If $k \leq l$, then $f_l f_{kl} = f_k$.
3. If N is another L-structure for which there are homomorphisms $g_k : M_k \to N$ such that $g_l f_{kl} = g_k$ whenever $k \leq l$, then there is a unique homomorphisms $g : M \to N$ such that $gf_k = g_k$ for all $k \in K$ ("universal property").
4. If all maps f_{kl} are embeddings, then so are all f_k.
5. If all maps f_{kl} are elementary embeddings, then so are all f_k ("elementary system lemma").

Proof.

Exercise! □