CHAPTER 1

New models from old

1. Directed systems

DEFINITION 1.1. A partially ordered set $(K, \leq)$ is called *directed*, if K is non-empty and for any two elements $x, y \in K$ there is an element $z \in K$ such that $x \leq z$ and $y \leq z$.

Note that non-empty linear orders (*aka* chains) are always directed.

DEFINITION 1.2. A *directed system of L-structures* consists of a family $(M_k)_{k \in K}$ of L-structures indexed by a directed partial order K, together with homomorphisms $f_{kl} \colon M_k \to M_l$ for $k \leq l$, satisfying:

- f_{kk} is the identity homomorphism on M_k,
- if $k \leq l \leq m$, then $f_{km} = f_{lm} f_{kl}$.

If K is a chain, we call $(M_k)_{k \in K}$ a *chain of L-structures*

If we have a directed system, then we can construct its *colimit*, another L-structure M with homomorphisms $f_k \colon M_k \to M$. To construct the underlying set of the model M, we first take the disjoint union of all the universes:

$$\sum_{k \in K} M_k = \{(k, a) \colon k \in K, a \in M_k\},$$

and then we define an equivalence relation on it:

$$(k, a) \sim (l, b) \colon \Leftrightarrow (\exists m \geq k, l)\, f_{km}(a) = f_{lm}(b).$$

The underlying set of M will be the set of equivalence classes, where denote the equivalence class of (k, a) by $[k, a]$.

M has an L-structure: if R is a relation symbol in L, we put

$$R^M([k_1, a_1], \ldots, [k_n, a_n])$$

if there is a $k \geq k_1, \ldots, k_n$ such that

$$(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n)) \in R^{M_k}.$$

And if g is a function symbol in L, we put

$$g^M([k_1, a_1], \ldots, [k_n, a_n]) \quad = \quad [k, g^{M_k}(f_{k_1 k}(a_1), \ldots, f_{k_n k}(a_n))],$$

where k is an element $\geq k_1, \ldots, k_n$. (Check that this makes sense!) In addition, the homomorphisms $f_k \colon M_k \to M$ are obtained by sending a to $[k, a]$.

The following theorem collects the most important facts about colimits of directed systems. Especially useful is part 5, often called the *elementary system lemma*.

THEOREM 1.3. (1) *All f_k are homomorphisms.*
(2) *If $k \leq l$, then $f_l f_{kl} = f_k$.*
(3) *If N is another L-structure for which there are homomorphisms $g_k \colon M_k \to N$ such that $g_l f_{kl} = g_k$ whenever $k \leq l$, then there is a unique homomorphisms $g \colon M \to N$ such that $g f_k = g_k$ for all $k \in K$ (this is the universal property of the colimit).*
(4) *If all maps f_{kl} are embeddings, then so are all f_k.*
(5) *If all maps f_{kl} are elementary embeddings, then so are all f_k.*

PROOF. Exercise! □

2. Ultraproducts

DEFINITION 1.4. Let I be a set. A collection $\mathcal{F}$ of subsets of I is called a *filter* (on I) if:

(1) $I \in \mathcal{F}, \emptyset \notin \mathcal{F}$;
(2) whenever $A, B \in \mathcal{F}$, then also $A \cap B \in \mathcal{F}$;
(3) whenever $A \in \mathcal{F}$ and $A \subseteq B$, then also $B \in \mathcal{F}$.

A filter which is maximal in the inclusion ordering is called an *ultrafilter*.

LEMMA 1.5. *A filter $\mathcal{U}$ is an ultrafilter iff for any $X \subseteq I$ either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$.*

PROOF. $\Rightarrow$: Let $\mathcal{U}$ be a maximal filter and suppose X is a set such that $X \notin \mathcal{U}$. Put

$$\mathcal{F} = \{Y \subseteq I : (\exists F \in \mathcal{U})\, F \cap X \subseteq Y\}.$$

Since $\mathcal{U} \subseteq \mathcal{F}$ and $X \in \mathcal{F}$, the set $\mathcal{F}$ cannot be filter; since it has all other properties of a filter, we must have $\emptyset \in \mathcal{F}$. So there is an element $F \in \mathcal{U}$ such that $F \cap X = \emptyset$ and hence $F \subseteq I \setminus X \in \mathcal{U}$.

$\Leftarrow$: Suppose $\mathcal{U}$ is a filter and for any $X \subseteq I$ either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$. If $\mathcal{U}$ would not be maximal, there would be a filter $\mathcal{F}$ extending $\mathcal{U}$. This would mean that there would be a subset $X \subseteq I$ such that $X \in \mathcal{F}$ and $X \notin \mathcal{U}$. But the latter implies that $I \setminus X \in \mathcal{U} \subseteq \mathcal{F}$. So $\emptyset = X \cap (I \setminus X) \in \mathcal{F}$, contradicting the fact that $\mathcal{F}$ is a filter. □

DEFINITION 1.6. For any element $i \in I$, the set $\{X \subseteq I : i \in X\}$ is an ultrafilter; ultrafilters of this form are called *principal*, the others are called *non-principal*.

If I is a finite set, then every ultrafilter on I is principal. If I is infinite, then there are non-principal ultrafilters. In fact, if I is infinite, then $\mathcal{F} = \{\, X \subseteq I : I \setminus X \text{ is finite} \,\}$ is a filter on I (this is the *Fréchet filter* on I). Since, by Zorn's Lemma, every filter can be extended to an ultrafilter, there is an ultrafilter $\mathcal{U} \supseteq \mathcal{F}$; such an ultrafilter has to be non-principal.

Now suppose we have a collection $\{M_i : i \in I\}$ of L-structures and $\mathcal{F}$ is a filter on I. We can construct a new L-structure M, as follows. Its universe is

$$\prod_{i \in I} M_i = \{f \colon I \to \bigcup_i M_i : (\forall i \in I)\, f(i) \in M_i\},$$

quotiented by the following equivalence relation:

$$f \sim g \quad :\Leftrightarrow \quad \{i \in I : f(i) = g(i)\} \in \mathcal{F}.$$

In addition, if g is an n-ary function symbol belonging to L and $[f_1], \ldots, [f_n] \in M$, then

$$g^M([f_1], \ldots, [f_n]) = [i \mapsto g^{M_i}(f_1(i), \ldots, f_n(i))],$$

and if R is an n-ary relation symbol belonging to L and $[f_1], \ldots, [f_n] \in M$, then

$$([f_1], \ldots, [f_n]) \in R^M \quad :\Leftrightarrow \quad \{\, i \in I \colon (f_1(i), \ldots, f_n(i)) \in R^{M_i}\,\} \in \mathcal{F},$$

where one should check, once again, that everything is well-defined. The resulting structure is denoted by $\prod M_i/\mathcal{F}$. We will be most interested in the special case where $\mathcal{F}$ is an ultrafilter, in which case $\prod M_i/\mathcal{F}$ is called an *ultraproduct*.

THEOREM 1.7. (Łoś's Theorem) *Let $\{M_i \colon i \in I\}$ be a collection of L-structures and $\mathcal{U}$ be an ultrafilter on I. Then we have for any formula $\varphi(x_1, \ldots, x_n)$ and $[f_1], \ldots, [f_n] \in \prod M_i/\mathcal{U}$ that*

$$\prod M_i/\mathcal{U} \models \varphi([f_1], \ldots, [f_n]) \quad \Leftrightarrow \quad \{i \in I \colon M_i \models \varphi(f_1(i), \ldots, f_n(i))\} \in \mathcal{U}.$$

PROOF. Exercise! □

COROLLARY 1.8. *If all M_i are models of some theory T, then so is $\prod M_i/\mathcal{U}$.*

COROLLARY 1.9. *Let M be an L-structure and $\mathcal{U}$ be an ultrafilter on a set I. Put $M_i = M$ and $M^* = \prod_{i \in I} M_i/\mathcal{U}$. Then the map $d \colon M \to M^*$ obtained by sending m to $[i \mapsto m]$ is an elementary embedding. If $|M| \geq |I|$ and $\mathcal{U}$ is non-principal, then this embedding is proper.*

Ultraproducts taken over a constant indexed family of models are called *ultrapowers*. In particular, the structure M^* in Corollary 1.9 is an ultrapower of M.

3. Additional exercises

EXERCISE 1. Do Exercise 2.5.20 in Marker.

CHAPTER 2

Preservation theorems

1. Characterisation universal theories

DEFINITION 2.1. A sentence is *universal* if it starts with a string of universal quantifiers followed by a quantifier-free formula. A theory is *universal* if it consists of universal sentences. A theory has a *universal axiomatisation* if it has the same class of models as a universal theory in the same language.

THEOREM 2.2. (The Łoś-Tarski Theorem) *T has a universal axiomatisation iff models of T are closed under substructures.*

PROOF. It is easy to see that models of a universal theory are closed under substructures, so we concentrate on the other direction. So let T be a theory such that its models are closed under substructures. Write

$$T_\forall = \{\varphi \colon T \models \varphi \text{ and } \varphi \text{ is universal }\}.$$

Clearly, $T \models T_\forall$. We need to prove the converse.

So suppose M is a model of $T_\forall$. Now it suffices to show that $T \cup \text{Diag}(M)$ is consistent. Because once we do that, it will have a model N. But since N is a model of $\text{Diag}(M)$, it will be an extension of M; and because N is a model of T and models of T are closed under substructures, M will be a model of T.

So the theorem will follow from the following claim: if $M \models T_\forall$, then $T \cup \text{Diag}(M)$ is consistent. *Proof of claim:* Suppose not. Then, by the compactness theorem, there are literals $\psi_1, \ldots, \psi_n \in \text{Diag}(M)$ which are inconsistent with T. Replace the constants from M in $\psi_1, \ldots, \psi_n$ by variables $x_1, \ldots, x_n$ and we obtain $\psi_1', \ldots, \psi_n'$; because the constants from M do not appear in T, the theory T is already inconsistent with $\exists x_1, \ldots, x_n\, (\, \psi_1' \wedge \ldots, \wedge \psi_n' \,)$. So $T \models \neg\exists x_1, \ldots, x_n\, (\, \psi_1' \wedge \ldots \psi_n' \,)$ and hence $T \models \forall x_1, \ldots, x_n\, (\, \neg(\psi_1' \wedge \ldots \psi_n') \,)$. Since M is a model of $T_\forall$, it follows that $M \models \forall x_1, \ldots, x_n\, (\, \neg(\psi_1' \wedge \ldots \psi_n') \,)$. On the other hand, $M \models \exists x_1, \ldots, x_n\, (\, \psi_1' \wedge \ldots, \wedge \psi_n' \,)$, since $\psi_1, \ldots, \psi_n \in \text{Diag}(M)$. Contradiction. □

2. Chang-Łoś-Suszko Theorem

DEFINITION 2.3. A $\forall\exists$-*sentence* is a sentence which consists first of a sequence of universal quantifiers, then a sequence of existential quantifiers and then a quantifier-free formula. A theory T can be axiomatised by $\forall\exists$-sentences if there is a set T' of $\forall\exists$-sentences such that T and T' have the same models.

DEFINITION 2.4. A theory T is *preserved by directed unions* if, for any directed system consisting of models of T and embeddings between them, also the colimit is a model T. And T is *preserved by unions of chains* if, for any chain of models of T and embeddings between them, also the colimit is a model of T.

THEOREM 2.5. (The Chang-Łoś-Suszko Theorem) *The following statements are equivalent:*

(1) T *is preserved by directed unions.*
(2) T *is preserved by unions of chains.*
(3) T *can be axiomatised by* $\forall\exists$*-sentences.*

PROOF. It is easy to see that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold, so we concentrate on (2) $\Rightarrow$ (3).

So suppose T is preserved by unions of chains. Again, let

$$T_{\forall\exists} = \{\varphi \colon \varphi \text{ is a } \forall\exists\text{-sentence and } T \models \varphi\},$$

and let B be a model of $T_{\forall\exists}$. We will construct a chain of embeddings

$$B = B_0 \to A_0 \to B_1 \to A_1 \to B_2 \to A_2 \ldots$$

such that:

(1) Each A_n is a model of T.
(2) The composed embeddings $B_n \to B_{n+1}$ are elementary.
(3) Every universal sentence in the language L_{B_n} true in B_n is also true in A_n (when regarding A_n is an L_{B_n}-structure via the embedding $B_n \to A_n$).

This will suffice, because when we take the colimit of the chain, then it is:

- the colimit of the A_n, and hence a model of T, by assumption on T.
- the colimit of the B_n, and hence elementary equivalent to each B_n.

So B is a model of T, as desired.

Construction of A_n: We need A_n to be a model of T and must have that every universal sentence in the language L_{B_n} true in B_n is also true in A_n. So let

$$T' = T \cup \{\varphi \colon \varphi \text{ is a universal } L_{B_n}\text{-formula and } B_n \models \varphi\};$$

we want to show that T' is consistent. Suppose not. Then, by compactness, there is a single universal sentence $\forall \overline{x}\, \varphi(\overline{x}, \overline{b})$ with $\overline{b} \in B_n$ and $B_n \models \forall \overline{x}\, \varphi(\overline{x}, \overline{b})$ that is already inconsistent with T. So

$$T \models \exists \overline{x}\ \neg\varphi(\overline{x}, \overline{b})$$

and

$$T \models \forall \overline{y}\, \exists \overline{x}\, \neg\varphi(\overline{x}, \overline{y})$$

because the b_i do not occur in T. Since $B_n \models T_{\forall\exists}$, we should have $B_n \models \forall \overline{y}\, \exists \overline{x}\, \neg\varphi(\overline{x}, \overline{y})$. But this contradicts the fact that $B_n \models \forall \overline{x}\, \varphi(\overline{x}, \overline{b})$.

Construction of B_{n+1}: We need $A_n \to B_{n+1}$ to be an embedding and $B_n \to B_{n+1}$ to be elementary. So let

$$T' = \mathrm{Diag}(A_n) \cup \mathrm{Diag}_{\mathrm{el}}(B_n)$$

(identifying the element of B_n with their image along the embedding $B_n \to A_n$); we want to show that T' is consistent. Suppose not. Then, by compactness, there is a quantifier-free sentence

$$\varphi(\overline{b}, \overline{a})$$

with $b_i \in B_n$ and $a_i \in A_n \setminus B_n$ which is true in A_n, but is inconsistent with $\mathrm{Diag}_{\mathrm{el}}(B_n)$. Since the a_i do not occur in B_n, we must have

$$B_n \models \forall \overline{x}\, \neg\varphi(\overline{b}, \overline{x}).$$

This contradicts the fact that all universal L_{B_n}-sentences true in B_n are also true in A_n. $\square$

3. Exercises

Exercise 2. Does the theory of fields have a universal axiomatisation?

Exercise 3. Prove: a theory has an existential axiomatisation iff its models are closed under extensions.

CHAPTER 3

The theorems of Robinson, Craig and Beth

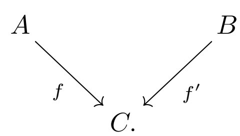
1. Robinson's Consistency Theorem

The aim of this section is to prove the statement:

> (Robinson's Consistency Theorem) Let L_1 and L_2 be two languages and $L = L_1 \cap L_2$. Suppose T_1 is an L_1-theory, T_2 an L_2-theory and both extend a complete L-theory T. If both T_1 and T_2 are consistent, then so is $T_1 \cup T_2$.

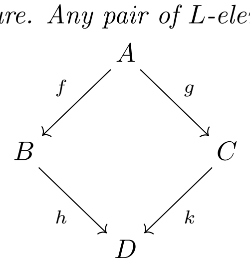
We first treat the special case where $L_1 \subseteq L_2$.

LEMMA 3.1. *Let $L \subseteq L'$ be languages and suppose A is an L-structure and B is an L'-structure. Suppose moreover $A \equiv B \upharpoonright L$. Then there is an L'-structure C and a diagram of elementary embeddings (f in L and f' in L')*



PROOF. Consider $T = \mathrm{Diag}^{L}_{\mathrm{el}}(A) \cup \mathrm{Diag}^{L'}_{\mathrm{el}}(B)$ (making sure we use different constants for the elements from A and B!). We need to show T has a model; so suppose T is inconsistent. Then, by compactness, a finite subset of T has no model; taking conjunctions, we have sentences $\varphi(\overline{a}) \in \mathrm{Diag}_{\mathrm{el}}(A)$ and $\psi(\overline{b}) \in \mathrm{Diag}_{\mathrm{el}}(B)$ that are contradictory. But as the a_j do not occur in L'_B, we must have that $B \models \neg\exists\overline{x}\,\varphi(\overline{x})$. This contradicts $A \equiv B \upharpoonright L$. $\square$

LEMMA 3.2. *Let $L \subseteq L'$ be languages, suppose A and B are L-structures and C is an L'-structure. Any pair of L-elementary embeddings $f: A \to B$ and $g: A \to C$ fit into a commuting square*

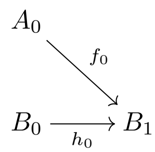


where D is an L'-structure, h is an L-elementary embedding and k is an L'-elementary embedding.

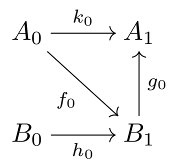
PROOF. Without loss of generality we may assume that L contains constants for all elements of A. Then simply apply Lemma 3.1. $\square$

THEOREM 3.3. (Robinson's Consistency Theorem) *Let L_1 and L_2 be two languages and $L = L_1 \cap L_2$. Suppose T_1 is an L_1-theory, T_2 an L_2-theory and both extend a complete L-theory T. If both T_1 and T_2 are consistent, then so is $T_1 \cup T_2$.*

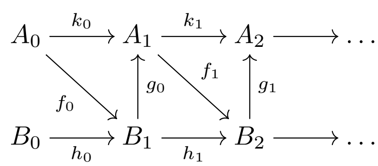
PROOF. Let A_0 be a model of T_1 and B_0 be a model of T_2. Since T is complete, their reducts to L are elementary equivalent, so, by the first lemma, there is a diagram



with h_0 an L_2-elementary embedding and f_0 an L-elementary embedding. Now by applying the second lemma to f_0 and the identity on A_0, we obtain



where g_0 is L-elementary and k_0 is L_1-elementary. Continuing in this way we obtain a diagram



where the k_i are L_1-elementary, the f_i and g_i are L-elementary and the h_i are L_2-elementary. The colimit C of this directed system is both the colimit of the A_i and of the B_i. So A_0 and B_0 embed elementarily into C by the elementary systems lemma; hence C is a model of both T_1 and T_2, as desired. □

2. Craig Interpolation

THEOREM 3.4. *Let φ and ψ be sentences in some language such that $\varphi \models \psi$. Then there is a sentence θ, a "(Craig) interpolant", such that*

(1) *$\varphi \models \theta$ and $\theta \models \psi$;*
(2) *every predicate, function or constant symbol that occurs in θ occurs also in both φ and ψ.*

PROOF. Let L be the common language of φ and ψ. We will show that $T_0 \models \psi$ where $T_0 = \{\sigma : \sigma \text{ is an } L\text{-sentence and } \varphi \models \sigma\}$. Let us first check that this suffices for proving the theorem: for then there are $\theta_1, \ldots, \theta_n \in T_0$ such that $\theta_1, \ldots, \theta_n \models \psi$ by compactness. So $\theta := \theta_1 \wedge \ldots \wedge \theta_n$ is an interpolant.

Claim: If $\varphi \models \psi$, then $T_0 \models \psi$ where $T_0 = \{\sigma \in L : \varphi \models \sigma\}$ and L is the common language of φ and ψ. *Proof of claim:* Suppose not. Then $T_0 \cup \{\neg\psi\}$ has a model A. Write $T = \mathrm{Th}_L(A)$. Observe that we now have $T_0 \subseteq T$ and:

(1) T is a complete L-theory.

(2) $T \cup \{\neg\psi\}$ is consistent (because A is a model).
(3) $T \cup \{\varphi\}$ is consistent. (*Proof:* Suppose not. Then, by the compactness theorem, there would a sentence $\sigma \in T$ such that $\varphi \models \neg\sigma$. But then $\neg\sigma \in T_0 \subseteq T$. Contradiction!)

This means we can apply Robinson's Consistency Theorem to deduce that $T \cup \{\neg\psi, \varphi\}$ is consistent. But that contradicts $\varphi \models \psi$. □

3. Beth Definability Theorem

DEFINITION 3.5. Let L be a language a P be a predicate symbol not in L, and let T be an $L \cup \{P\}$-theory. T *defines* P *implicitly* if any L-structure M has at most one expansion to an $L \cup \{P\}$-structure which models T. There is another way of saying this: let T' be the theory T with all occurrences of P replaced by P', another predicate symbol not in L. Then T *defines* P *implicitly* iff

$$T \cup T' \models \forall x_1, \dots x_n \left(P(x_1, \dots, x_n) \leftrightarrow P'(x_1, \dots, x_n) \right).$$

T *defines* P *explicitly*, if there is an L-formula $\varphi(x_1, \dots, x_n)$ such that

$$T \models \forall x_1, \dots, x_n \left(P(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n) \right).$$

THEOREM 3.6. (Beth Definability Theorem) *T defines P implicitly if and only if T defines P explicitly.*

PROOF. It is easy to see that T defines P implicitly in case T defines P explicitly. So we prove the other direction.

Suppose T defines P implicitly. Add new constants $c_1, \dots, c_n$ to the language. Then we have

$$T \cup T' \models P(c_1, \dots, c_n) \to P'(c_1, \dots, c_n).$$

Using compactness and taking conjunctions we can find an $L \cup \{P\}$-formula ψ such that $T \models \psi$ and

$$\psi \wedge \psi' \models P(c_1, \dots, c_n) \to P'(c_1, \dots, c_n)$$

(where ψ' is ψ with all occurrences of P replaced by P'). Taking all the Ps to one side and the P's to another, we get

$$\psi \wedge P(c_1, \dots, c_n) \models \psi' \to P'(c_1, \dots, c_n)$$

So there is a Craig interpolant θ in the language $L \cup \{c_1, \dots, c_n\}$ such that

$$\psi \wedge P(c_1, \dots, c_n) \models \theta \text{ and } \theta \models \psi' \wedge P'(c_1, \dots, c_n)$$

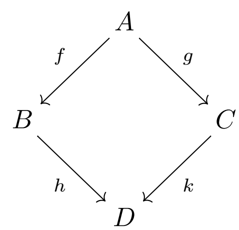
By symmetry also

$$\psi' \wedge P'(c_1, \dots, c_n) \models \theta \text{ and } \theta \models \psi \wedge P(c_1, \dots, c_n)$$

So $\theta = \theta(c_1, \dots, c_n)$ is, modulo T, equivalent to $P(c_1, \dots, c_n)$; hence $\theta(x_1, \dots, x_n)$ defines P explicitly. □

4. Exercises

Exercise 4. Use Robinson's Consistency Theorem to prove the following Amalgamation Theorem: Let L_1, L_2 be languages and $L = L_1 \cap L_2$, and suppose A, B and C are structures in the languages L, L_1 and L_2, respectively. Any pair of L-elementary embeddings $f\colon A \to B$ and $g\colon A \to C$ fit into a commuting square



where D is an $L_1 \cup L_2$-structure, h is an L_1-elementary embedding and k is an L_2-elementary embedding.