Homework 3, Exercise 1, Proof of Claim

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Claim 1. *If the theory $T_n \cup \{\exists x \psi(x)\} \cup T \cup \mathsf{ElDiag}(M)$ is satisfiable, then also the theory $T_n \cup \{\psi(m)\} \cup T \cup \mathsf{ElDiag}(M)$ is satisfiable for some $m \in M$.*

Proof. Let N be a model of $T_n \cup \{\exists x \psi(x)\} \cup T \cup \mathsf{ElDiag}(M)$. Consider the partial type

$$p(x) := \{\phi(x) \in L_A \mid T_n \cup T \cup \mathsf{ElDiag}(M) \vDash \forall x(\psi(x) \to \phi(x))\}$$

where $A \subseteq M$ is the set of constants occurring in T_n. Note that $|A| < \omega$.

By $N \vDash \exists x \psi(x)$ there is an $n \in N$ such that n realizes p in N.

We now show that p is also finitely realized in M. Note that p is closed under conjunction. Hence w.l.o.g. we take a $\phi \in p$. Then $N \vDash \phi(n)$ and therefore $N \vDash \exists x \phi(x)$. The latter is an L_A sentence and by $N \vDash \mathsf{ElDiag}(M)$ we have that $N \equiv_{L_M} M$, hence also $N \equiv_{L_A} M$ and therefore $M \vDash \exists x \phi(x)$.

As ϕ was arbitrary this shows that p is finitely realized in M. Because M is ω-saturated p is also realized in M. Let $m \in M$ be an element that realizes p in M.

Now suppose that $T_n \cup \{\psi(m)\} \cup T \cup \mathsf{ElDiag}(M)$ is not satisfiable. Then by compactness there is an L_M sentence $\theta \in \mathsf{ElDiag}(M)$ such that $T_n \cup T \vDash \psi(m) \to \neg\theta$.

We can write θ as an L-formula with additional parameters m itself, $\vec{a}$ from $A \setminus \{m\}$ and $\vec{m'}$ from $(M \setminus A) \setminus \{m\}$. Then $T_n \cup T \vDash \psi(m) \to \neg\theta(m, \vec{a}, \vec{m'})$. Note that the $\vec{m'}$ are not in T or T_n, hence $T_n \cup T \vDash \psi(m) \to \forall \vec{y} \neg\theta(m, \vec{a}, \vec{y})$. Now distinguish two cases:

1. Suppose $m \notin A$. Then also m does not occur in $T_n \cup T$ and we have $T_n \cup T \vDash \forall(x)(\psi(x) \to \forall \vec{y} \neg\theta(x, \vec{a}, \vec{y}))$. Note that the consequent is an L_A formula. Hence by definition of p we have that $\forall \vec{y} \neg\theta(x, \vec{a}, \vec{y}) \in p$.

   But m realizes p, hence $M \vDash \forall \vec{y} \neg\theta(m, \vec{a}, \vec{y})$ and in particular $M \vDash \neg\theta(m, \vec{a}, \vec{m'})$.

2. Suppose $m \in A$. By compactness there is a $\chi(m, \vec{a})$ such that $T_n \vDash \chi(m, \vec{a})$ and

   $$T \vDash \psi(m) \to \forall \vec{y} \neg(\theta(m, \vec{a}, \vec{y}) \wedge \chi(m, \vec{a})).$$

   Note that m does not occur in T, hence

   $$T \vDash \forall x(\psi(x) \to \forall \vec{y} \neg(\theta(x, \vec{a}, \vec{y}) \wedge \chi(m, \vec{a})))$$

   Again the consequent is an L_A formula, hence $\forall \vec{y} \neg(\theta(x, \vec{a}, \vec{y}) \wedge \chi(m, \vec{a})) \in p$.

   We know that m realizes p, hence $M \vDash \forall \vec{y} \neg(\theta(m, \vec{a}, \vec{y}) \wedge \chi(m, \vec{a}))$. Instantiating $\vec{m'}$ for $\vec{y}$ we get $M \vDash \neg(\theta(m, \vec{a}, \vec{m'}) \wedge \chi(m, \vec{a}))$.

   But we also have that $T_n \vDash \chi(m, \vec{a})$, therefore $\chi(m, \vec{a}) \in p$. (This is not a typo: The type p also contains a lot of formulas not mentioning the free variable x.)

   Hence by boolean reasoning it must be that $M \vDash \neg(\theta(m, \vec{a}, \vec{m'})$.

In both cases we have a contradiction, because by $\theta(m, \vec{a}, \vec{m'}) \in \mathsf{ElDiag}(M)$ we also have $M \vDash \theta(m, \vec{a}, \vec{m'})$. Hence $T_n \cup \{\psi(m)\} \cup T \cup \mathsf{ElDiag}(M)$ has to be satisfiable. □