3rd Homework sheet Model Theory

- *There is a second exercise on the next page!*
- Deadline: 25 February.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture at 15:00.*
- Good luck!

Exercise 1 *(50 points)* Throughout this exercise M will be a countable ω-saturated model in a language L. Suppose moreover that L' is a countable language extending L and that T is an L'-theory which is consistent with $\mathrm{Th}(M)$. The aim of this exercise is to show that M can be expanded to an L'-structure M' with $M' \models T$. To this purpose let $(\varphi_n : n \in \mathbb{N})$ be an enumeration of all the L'_M-formulas.

(a) Show that there is an increasing sequence of L'_M-theories $(T_n \colon n \in \mathbb{N})$ such that:

   (i) each $T_n \cup T \cup \mathrm{ElDiag}(M)$ is satisfiable.

   (ii) either $\varphi_n \in T_{n+1}$ or $\neg\varphi_n \in T_{n+1}$.

   (iii) if $\varphi_n \in T_{n+1}$ and φ_n is of the form $\exists x\, \psi(x)$, then $\psi(m) \in T_{n+1}$ for some $m \in M$.

   Hint: Construct such theories T_n by recursion, starting with $T_0 = \emptyset$. When building T_{n+1} from T_n, you have to make sure that (ii) and (iii) hold. The difficult bit is to show that in (iii) one can choose the witness m from M itself: it is here that ω-saturation is used.

(b) Let $(T_n \colon n \in \mathbb{N})$ be a sequence of theories as in (a) and put $T' = \bigcup_{n\in\mathbb{N}} T_n$. Build a model N of T' as on page 3 of the slides for week 2. Show that the L-reduct of N is isomorphic to M, so that N can be regarded as the desired expansion of M that also models T.

Exercise 2 *(50 points)* In this exercise n is a fixed natural number and κ is a fixed infinite cardinal. Suppose that T is a theory in a language L for which the type space $S_n(T)$ has at most κ many points. Prove that there are, up to logical equivalence over T, at most κ many L-formulas with free variables among $x_1, \ldots, x_n$.