7th Exercise sheet Model Theory
18 Mar 2015

Exercise 1 Let L be the first-order language of linear orderings.

(a) Show that if $h \leq 2^k$ then there is a formula $\varphi(x, y)$ of L of quantifier depth $\leq k$ which expresses (in any linear ordering) "$x < y$ and the distance between x and y is at least h".

(b) Let $M = (\mathbb{Z}, <)$. Show that if $a_0 < \ldots < a_{n-1}$ and $b_0 < \ldots < b_{n-1}$ in M, then $(M, \overline{a}) \equiv_k (M, \overline{b})$ if and only if for all $m < n - 1$ and all $i \leq 2^k$,

$$\operatorname{dist}(a_m, a_{m+1}) = i \Leftrightarrow \operatorname{dist}(b_m, b_{m+1}) = i.$$

Exercise 2 Show that there is no formula of first-order logic which expresses "(a, b) is in the transitive closure of R", even on finite structures. (For infinite structures it is easy to show that there is no such formula; why?)

Exercise 3 Let M be an L-structure and $\mathcal{U}$ be an ultrafilter on an infinite set I. Put $M_i = M$ and $M^* = \prod_{i \in I} M_i / \mathcal{U}$.

(a) Define a map $d: M \to M^*$ by sending m to the equivalence class of the function which is constant m. Show that d is an elementary embedding.

(b) Prove that if $|M| \geq |I|$ and $\mathcal{U}$ is non-principal, then the embedding d from (a) cannot be surjective.

Exercise 4 In this exercise $I = \mathbb{N}$ and $\mathcal{U}$ is a non-principal ultrafilter on the set of natural numbers. Suppose that for each $i \in I$ we have a model M_i, each in the same countable language. Write $M^* = \prod_{i \in I} M_i / \mathcal{U}$. The aim of this exercise is to show that M^* is ω-saturated.

So let $A = \{[f_1], \ldots, [f_k]\}$ be a collection of k-many elements from M^*, and let $p(x)$ be a partial 1-type with parameters from A. Since the language is countable we can enumerate the formulas $(\varphi_j : j \in \mathbb{N})$ in $p(x)$ and, by taking conjunctions, we may assume, without loss of generality, that $\varphi_{j+1}(x) \to \varphi_j(x)$. Instead of $\varphi_j(x)$ we will write $\theta_j(x, [f_1], \ldots, [f_k])$ where θ_j is an L-formula.

(a) Let

$$D_j = \{\, i \in \mathbb{N} \colon M_i \models \exists x\, \theta_i(x, f_1(i), \ldots, f_k(i)) \,\}.$$

Show that $D_j \in \mathcal{U}$.

(b) Find a $g \in \prod_{i \in I} M_i$ such that if $j \leq i$ and $i \in D_j$, then

$$M_i \models \theta_j(g(i), f_1(i), \ldots, f_k(i)).$$

(c) Show that $[g]$ realizes $p(x)$. Where do you use the fact that $\mathcal{U}$ is non-principal? Conclude that M^* is ω-saturated.