6th Homework sheet Category Theory

- Deadline: 23 May, 13:00 sharp.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture*.
- Good luck!

Exercise 1 *(10 points)* A poset P is *complete* if every subset $X \subseteq P$ has a least upper bound and a greatest lower bound (equivalently, it is a poset which is both complete and cocomplete as a category). This means in particular that it has a least and greatest element. A *complete Heyting algebra* is a complete poset which is also cartesian closed as a category. Throughout this exercise H will be a fixed complete Heyting algebra.

An *H-set* is a pair (A, α) where A is a set and α is a function $A \to H$. A morphism of H-sets $f\colon (B, \beta) \to (A, \alpha)$ is a function $f\colon B \to A$ such that

$$\beta(b) \leq \alpha(f(b))$$

for every $b \in B$. Since such functions are closed under composition and contain all identities, this defines a category which we will denote by $H\mathbf{Sets}$.

You have already proved in an earlier homework sheet that this category has pullbacks, so any morphism $f\colon (B, \beta) \to (A, \alpha)$ in this category induces a monotone function

$$f^*\colon \mathrm{Sub}(A, \alpha) \to \mathrm{Sub}(B, \beta)$$

by pullback. Show that this operation has a left adjoint $\exists_f$ and a right adjoint $\forall_f$.