2nd Homework sheet Category Theory

- Deadline: 25 April, 13:00 sharp.
- Submit your solutions by handing them to the lecturer at the *beginning of the lecture*.
- Good luck!

Exercise 1 Throughout this exercise M is a fixed monoid with unit element 1 and multiplication $\cdot_M \colon M \times M \to M$. If X is a set, then an *action* of M on X is a function $\cdot_X \colon M \times X \to X$ satisfying the following equalities

$$\begin{aligned} 1 \cdot_X x &= x \\ m \cdot_X (m' \cdot_X x) &= (m \cdot_M m') \cdot_X x \end{aligned}$$

for all $m, m' \in M$ and $x \in X$. A set X together with an action of M is called an *M-set*. If $(X, \cdot_X)$ and $(Y, \cdot_Y)$ are M-sets, then a function $f \colon X \to Y$ is a morphism of M-sets, if

$$m \cdot_Y f(x) = f(m \cdot_X x)$$

for every $m \in M$ and $x \in X$. The resulting category of M-sets is denoted $M\mathbf{Sets}$.

(a) *(5 points)* Show that the category $M\mathbf{Sets}$ has all small limits and colimits.

(b) *(5 points)* Let $U \colon M\mathbf{Sets} \to \mathbf{Sets}$ be the forgetful functor sending $(X, \cdot_X)$ to X. Show that there is a functor $F \colon \mathbf{Sets} \to M\mathbf{Sets}$ which assigns to every set X "the free M-set on X": that is, FX is an M-set equipped with a function $i_X \colon X \to UFX$ such that for any M-set Y and function $f \colon X \to UY$ there exists a unique morphism of M-sets $\overline{f} \colon FX \to Y$ such that $f = U\overline{f} \circ i_X$.